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How Intuitive Geometry Serves the Community*

By A. Brown Miller
West Technical High School, Cleveland, Ohio

As a rule, we are at first impressed by the possibility of mathematics serving the community commercially. We think of those situations in which the dollar sign is used. There are the banks, stores, places of entertainment, and so on, where we either pay for services or recreation, or receive money for goods, or services rendered. Before it is possible for many of these activities to exist, other forms of mathematics have been utilized in providing the buildings, streets, bridges, sanitation, light, and so forth. Beauty and orderly arrangement are desirable in all of these as well as comfort and convenience.

There was a time when towns as well as homes were constructed without regard to plan or amount of material. In some communities this is still the case, but in general that day has passed as is evidenced in the majority of cases by the beauty of arrangement and variety of materials used in urban developments, homes and public buildings. Now the amount of building material is computed accurately according to a plan before the construction is begun.

No doubt you are thinking that this is not an engineering convention. The major constructions do involve such problems, but there are many mathematical implications which fall within the field of intuitive geometry. The concepts of size, form, and position are involved in the whole problem.

All that appears in the set-up of a community is a direct result of the imagination, vision, skill, knowledge, and wealth of that community. Our responsibility as educators is not only to strive to perpetuate what we find that is good, and the practices and processes that are now in vogue, but as well to instigate the creation of new vision and to make new applications of the knowledge we possess.

It is important that pupils be encouraged to improve upon the appearance and methods of construction found in communities today. Necessarily the usual applications of mathematics will require the major part of the time devoted to its teaching. We will serve the community well if we train pupils in those skills which daily life requires and at the same time arouse their interest in the larger aspects of life about them which directly, or indirectly, influence them, and which in due time will be influenced by them.

The junior high school pupil is at a stage in his development in which he is greatly interested in the exploration of life and institutions. He is equally interested in all those forces which enter into providing his

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material environment. He is emotionally alert to form and beauty and is interested in their creation.

The automobile has made it possible for many of the children to become acquainted with communities other than their own. They are aware of the fact that communities differ. One of the things in common however is that each has some central point from which they radiate in some plan or order. There is a public square or business center. Any plan put upon paper would naturally begin there and streets and avenues of transportation radiate from it. In looking at such a plan, or map, a variety of geometric plane figures are evident. Not only can such a plan be used in teaching or further exemplifying plane figures, but other mathematical concepts can be recognized. A center or lines of reference are necessary to locate places of business or residence. In this relationship number gets a new significance, the points of the compass are used; E 15th St., N 5th Ave., and other designations. The study of the compass and the idea of position as related to lines or points of reference are phases of intuitive geometry. The use of number to locate offices in a large building, seats in a theater, houses on streets, when suggested to children, usually bring from them many more similar illustrations.

In studying the plan of most central squares in a town or city, there is usually revealed some formal arrangement where balance or symmetry, either point or line, is applied. Street intersections are frequently beautified by planting. The formal arrangement of such planting often assumes some geometric design. These plans offer interesting material for scale drawings and encourage the making of original plans for such use. In this way an interest is aroused in improving the appearance of these spots, in maintaining and protecting what is there, and possibly in creating a demand for developing something new for which citizens will be willing to pay, because they recognize how such a beauty spot would improve the community. The landscaping about all schools and other public buildings when critically studied, offers to the pupils real examples of desirable or undesirable appearance. A formal garden where a definite plan applying symmetry, balance, congruence, and similarity is developed, presents a pleasing appearance recognized by all.

Parks and recreational centers are becoming more and more a part of every community. In some places they have just grown up like Topsy. In other places they are where they are as the result of a definite civic plan. Where they have just grown up, a real service can be rendered by a careful study of the present situation. Have tennis courts and baseball diamonds the best orientation with respect to the sun? How accurately are they measured? Of lasting benefit to the pupils as well as to the community would be the activity of improving such a recreation center. This would involve measuring the whole area, followed by the making of a scale drawing of it as it is, and the careful planning of a new arrangement of the various places for sports, thus providing for a larger variety of activities which would serve more children of different ages. In any such effort there would occur many situations where making right angled corners would be necessary and could call into practical use the triangle with sides in the 3, 4, 5 relationship. A rope with 13 equally spaced knots would serve this purpose, or the carpenter's method of using a five-foot board placed three feet from the corner on one side and four feet from the same corner on the other side.

Architecture is another field in which the study of form and position serves the community. Analysis of the architecture of public buildings, schools, churches, and houses, is always interesting and profitable to children. They soon sense that form, balance, symmetry, congruence, etc., are present where there is a satisfying appearance. In buildings there are to be seen rectangular solids, prisms, cylinders, cones,

pyramids, hemispheres, rectangles, squares, hexagons, circles, semicircles, arcs, angles of various kinds, in a harmonic whole suitable to its purpose. Time can profitably be spent in this study, as impressions thus formed will influence the architecture of future buildings. Magazines and other sources of fine illustrations of architecture make possible the development of a bulletin board in the classroom, which will further influence these impressions.

In many communities W.P.A. projects are now in progress, or have been completed. What have they added in appearance to the community? Has a community the right to place restrictions upon building plans? Is a community, or city architect a justifiable expense? What will the future citizen whom you are now influencing, demand?

The insulation of houses, to make them more comfortable in summer as well as in winter, is becoming a very important business. In presenting a contract for such work, the owner of the house is shown a plan, carefully drawn to scale by the engineer. An understanding of the simplest phases of intuitive geometry aids the owner in following the explanation given by the engineer as he points out the areas which will contain the insulation.

In estimating the cost of new linoleum for the kitchen floor, intuitive geometry is used in its measurement. It is also used in estimating the amount of material needed for other household furnishings.

A common problem in every community is that of supplying a sufficient supply of good water. Here the geometry of size serves the community. As a rule reservoirs, or water towers, are so located that pupils can obtain some approximate idea of their size, and thus find their approximate capacity. The vast quantity of water in a reservoir is invariably a great surprise. To interpret the relation between a cubic foot and its capacity in gallons, finding the capacity of a small aquarium is helpful.

Another interest of every community is the building of streets and roads. Some realization of the cost of such items can be gained by determining the cost of a private driveway if constructed of different types of material. Unit costs of paving and road-building can be obtained and used as problem material.

Much of the tax money is spent in construction of buildings, excavating, building roads and streets, providing water, light, and so forth. To have some intelligent understanding of the possible cost of these items, the citizen must have had experience with problems of size and capacity dealing with the simpler geometric forms, both plane and solid. Some idea of what it costs to heat a public building can be derived from a knowledge of the capacity of the school's coal bin and the average supply needed to heat the building for a definite period of time. An impression of the area of a recreation center can be gained by finding the area required for a tennis court, a baseball diamond and field.

Rents and taxes are frequently based upon the area of the floor space or the volume of buildings. The valuation of property is given as so much a foot-front. The cost of storage depends upon the number of cubic feet in the storage space. In department stores the several departments must produce an income commensurate with the floor space they occupy. In the warehouse there is charged against each department the space occupied by their goods.

Few people realize how much the geometric ideas; symmetry, congruence, equality, and similarity influence their lives. The automobile in which we ride has many applications of these principles. Before new models are produced the stamping dies must be constructed, which press into form sections of the body and many of its fittings. These are produced as congruent parts by simple repetition. If this were not true, a car would cost several times its present price. All standardized parts are examples of the application of congruence to industry. Ten-cent stores would not exist were it not for the practice of these

principles in industry. The difference in the cost of tailor-made and factory-made suits is the result of the application of congruence and similarity. Many congruent portions are made at a single cutting by machinery. Sizes 30 to 49 exemplifies similarity, the suit as a whole represents symmetry.

If a community is to appreciate the extent to which intuitive geometry serves its interests the teacher certainly must be aware of the many instances of the influence of geometric concepts. As he walks, rides, drives, and so forth, he observes the mathematical implications of his environment. May he be forgiven as his mind wanders from the sermon to a study of the arcs, angles, and geometric designs in the church. Mathematics taught by such a teacher cannot help but serve the community.

Not all teachers are by temperament adapted to interpret the problems of the community, for too many teachers problems exist only in books and are to be solved by rule. This is a decided handicap. Placing data in the hands of such a teacher is ineffective. As one engaged in training young teachers I have been greatly impressed by the wide range of degree to which imagination, ability to translate by illustration, a sense of the fitness of things, appreciation of the practical, are possessed by these prospective teachers. In the selection of a teacher of intuitive geometry these qualities are of paramount importance. To these I should add a natural civic interest.

The community as well as the pupil is served by the study of community problems to the degree to which he participates in research within that community. For this research to be of value, a great deal of careful planning with respect to the division of the responsibilities within the group must be made in advance. These responsibilities should have some relationship to the interests of the individual pupils. Research should be limited to those situations in which the mathematical implications are evident and need not be forced.

There need be no conflict between serving the child and serving the community. The best problems are always those which are near to the child's experience in the community. The careful building of the essential sequence of skills and concepts will at all times be the concern of any good teacher. The best incentive to a careful mastery of these skills and concepts is to keep the vital problems of application within the degree of difficulty of the skills developed.

While I feel that this attempt to teach intuitive geometry in such a way that it will serve the community is a worthy purpose, it will not challenge all pupils. They strangely manifest many of the characteristics of adults. In any adult undertaking, not all purposefully and actively participate. Not any one activity will enlist total participation. This does not condemn the activity. The value of the activity, however is reflected in the degree of interest aroused and the extent to which it ultimately serves him as a citizen of the community.

Intuitive geometry always will be of prime service to any community. Its future value lies in the vision created in our pupils by teachers aware of its possibilities.

The Mathematician

Stranger alike to traffic's clamor crude
And to joy's throbbing, intricate design,
He stands serene. A formula, a line,
With changeless beauty is by him endued.
Striver for truth's perfection, no light mood
May move him. Differential, axiom, sign,
Bring to him glimpses of the far divine,
Marking the boundaries of finitude.

By Euclid's theorems cramped, he seeks new spheres
And walks in high, far ways forever free,
Toils with awed vision through the ordered years
Till, from the all-but-handled harmony,
In some grave vision Deity appears,
And in a graph he finds Eternity.

NELSON ANTRIM CRAWFORD

Teaching Relationships in Junior High School

By H. C. Christofferson
Miami University, Oxford, Ohio

For many years, we have emphasized relationships in the teaching of intuitive geometry. We have developed formulas for finding the area of a rectangle and of a triangle. These express very concisely and clearly the relationship which exists between A and the variables b and h. The formula for the area of a circle, $A=\pi r^2$, is a bit more complex and I am inclined to believe that we have studied the formula without really appreciating the relationship involved. Let me emphasize by means of a little model the double-headed effect of r upon A.

Figure 1 represents a circle cut into 24 equal sectors with one half of them so spread out that the sectors of the other half would fit in to make the parallelogram-shaped figure shown. The base of the figure is a wavy line composed of twelve equal arcs and is equal in length to half the circumference of the circle. The altitude is the radius of the circle. If this figure had been made by dividing the circle into a million sectors, the waviness of the bases could not be seen by the human eye, the altitude would still be the length of the radius, and figure itself would be much more nearly a rectangle.

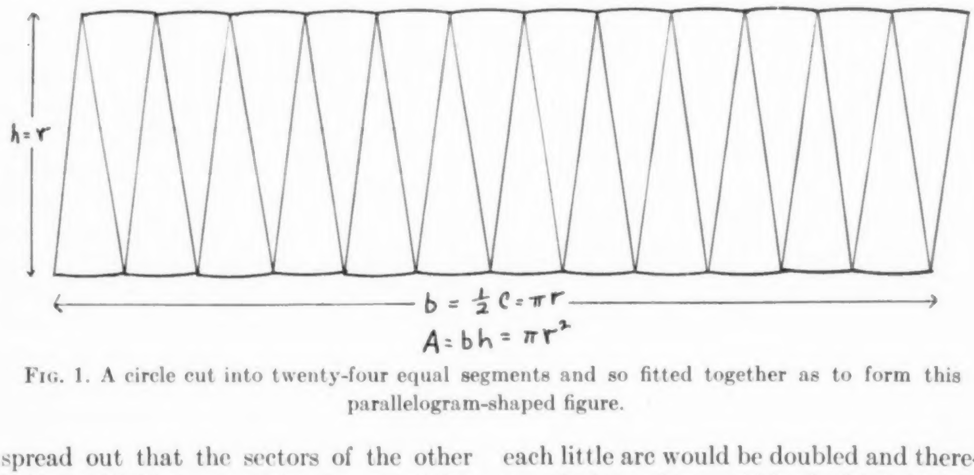

Fig. 1. A circle cut into twenty-four equal segments and so fitted together as to form this parallelogram-shaped figure.

Notice that the area of this parallelogram is equal to the base times the altitude. The base is half a circle, and since its whole circumference is $2\pi r$, half of its circumference is πr. The height is r, consequently the area is $\pi r \cdot r$ or $A=\pi r^2$. This illustrates unforgettably the relationship between the area and the radius of a circle. Now the question is, What happens if the radius changes? Would merely the height of this parallelogram be changed, or would also the length of the arc of each of these little sectors be increased?

Suppose the radius were doubled, it is evident that not only would the height be doubled in this parallelogram, but that each little arc would be doubled and therefore the base also would be doubled. Consequently, we have established a relationship that the area of the circle is doubly dependent upon the radius. Having once established this formula and the relationship which it expresses, we do not forever afterward force children to cut circles into sectors and go through all the laborious thinking that was involved in developing that formula. The purpose of a formula is to save time and to avoid error.

We do not, however, use a similarly sen-

sible method of generalizing for some of the other problems that we teach in junior high school mathematics. Let me take an illustration from eighth grade mathematics and another from ninth grade mathematics. Most books include the area of a circle in the seventh grade, thus the three illustrations together will be my defense for the subject, "Teaching Relationships in Junior High School."

The installment price of a refrigerator is \$170. The discount for cash is \$10, about 6%. 6% of \$170 = \$10.20. The cash price, therefore, is \$160. The installment plan is: cash \$50, and monthly payments of \$20 per month for 6 months. What is the rate of interest charged for the carrying of this account? It is evident that if the man purchased something for which the cash price is \$160, that after he paid \$50, he really owed \$110 during the first month. If he had borrowed \$110 from a bank, he could have paid cash for the machine.

His indebtedness may be summarized as follows:

1 month	\$110
1 month	90
1 month	70
1 month	50
1 month	30
1 month	10
T_b	= \$360
Av. Debt	= \$ 60

$$60\times\frac{6}{12}\times r=10$$

$$360r=120$$

$$r=\frac{1}{3}=33\tfrac{1}{3}\%$$

When he made his last \$20 payment, he paid the \$10 balance of his indebtedness and the \$10 extra which was charged for the privilege of deferred payments.

We do teach a formula in junior high school that $i=prt$. This emphasizes the relationship between interest and the principal, rate, and time. It is evident in this case that the interest is \$10, but the principal is not so easily determined. The principal is a changing amount, beginning with \$110 and ending with \$10. Probably the simplest way to think of that is to consider the average debt. He owed each of these amounts for one month, so the average amount would be the sum of these amounts divided by 6. The sum of these amounts is \$360, which divided by 6, makes the average debt \$60. Now we can think of the \$60 as the average debt which he owed during the 6 months. Therefore, $60\times 6/12\times r=10$. If we solve this for r, it is evident that $30r=10$. $r=\frac{1}{3}$ or $33\frac{1}{3}\%$ interest.

Traditionally, we have been content with *an* answer to problems of this kind. We have not emphasized the relationships which are involved in situations similar to this. Would we be similarly content with a crude analysis for the area of a circle as illustrated by the technique used in deriving its formula? Let us then proceed to analyze this installment buying situation to discover the relationships which are involved and to produce a simple formula similar to that for the area of a circle.

Where did this $33\frac{1}{3}\%$ come from? As we trace it back to the main equation, we recognize that the \$10 is the cash discount. This we can represent by the letter d. The \$60 is the total of the balances divided by the number of months. That we can represent by T_b/n. Six is the number of months, n, during which the account was unpaid and that might vary. Twelve, however, is the number of months in a year and does not vary. r is the rate for which we are solving. Now it is evident that the n's will cancel in this general equation, $T_b/n\times n/12\times r=d$, and if we solve this for r we get a very simple, beautiful relationship, $r=12d/T_b$. By analogy the corresponding formula in a weekly payment plan is, $r=52d/T_b$. However, even though we now have a formula, we have not yet adequately analyzed the relationships involved in this situation. What features of this situation would make r large and what features would make it small? It is evident that a large cash discount would make for a large rate of interest, or a small cash discount for a small rate of interest. That relationship is relatively simple. The denominator of this fraction, however, presents a portion of a situation which is not so simple. If the total of the balances were

large, then it is evident that the rate would be small. What features of any situation would make the total of the balances large or larger or smaller? Let us suppose that in place of \$20 a month for 6 months, payments had been made of \$10 a month for 12 months, or \$40 a month for 3 months. What would happen to the balances? During the first month it would be \$110 in either case. The following table summarizes the debt situation.

\$10 per Month		\$40 per Month	
1 month	\$110	1 month	\$110
1 month	100	1 month	70
1 month	90	1 month	30
1 month	80	T_b	=\$210
1 month	70		
1 month	60		
1 month	50		
1 month	40		
1 month	30		
1 month	20		
1 month	10		
1 month	0		
T_b	=\$660		

$$r=\frac{12d}{T_b}$$

$$r=\frac{12d}{T_b}$$

$$r_3=\frac{120}{210}=57.1\%$$

$$r_{12}=\frac{120}{660}=18.2\%$$

$$r_6=\frac{120}{360}=33\tfrac{1}{3}\%$$

During the twelfth month in the first case, he would have owed nothing, but at the end of that month, he would pay the \$10 charged for the privilege of deferred payments. It is evident that an increase in the time for payment will in this situation greatly increase the denominator of the fraction and therefore greatly decrease the rate. Further analysis of the relationship between T_b and n is very interesting, but is probably too complex for an eighth grader. However, an appreciation of the relationships which are revealed in the above formula is not too difficult for an eighth grader and I submit to you that the derivation of the formula for installment buying is far more important for an eighth grader than the formula for finding the area of a circle.

A third illustration, taken from ninth grade mathematics, reveals a surprisingly simple formula for mixture problems. A dairyman has 100 gallons of 3% milk and he has cream which is 40% butter fat. He wants a mixture for selling that is 4% butter fat. How much 40% cream shall he add to the 3% milk to make 4% milk? The traditional solution is one with which you are all familiar.

$$(1)\quad 3\% \text{ of } 100+40\% \text{ of } x=4\% \text{ of } (x+100)$$

$$(2)\quad 3+.40x=.04x+4$$

$$(3)\quad .36x=1$$

$$(4)\quad x=\frac{1}{.36}=\frac{100}{36}=\frac{25}{9}=2\ 7/9$$

Therefore, 2 7/9 gallons cream will do the trick.

Again we have *an* answer to a problem. Is the important thing in the education of children in 1941 the getting of answers to problems, or is it rather the discovering of relationships—the writing and determination of formulas which generalize that solution, the making of graphs which picture the relationship, the building of tables which display the relationship? Let us derive a formula which this dairyman can use with the ease and accuracy that a man can find the area of a circle. This can be done by substituting letters for the variable quantities: m for milk, p_m for the butter fat content of the original milk, c for gallons of cream, p_c for per cent of butter fat in the cream. Making those changes the equation (2) becomes:

$$(1)\quad p_m m+.04c=.04(c+m)=.04c+.04m$$

$$(2)\quad p_c c-.04c=.04m-p_m m$$

$$(3)\quad c(p_c-.04)=m(.04-p_m)$$

$$(4)\quad c=m\frac{.04-p_m}{p_c-.04}$$

Now let us make some other substitutions. This $.04-p_m$ is really the increase in the butter fat content of milk. Let us call that increase in per cent of butter fat in the milk, i_m. The denominator, p_c-4 is the decrease in the butter fat content in the cream, d_c.

If we also divide the equation (4) by m, we will have a very nice, simple relationship: $\frac{c}{m}=\frac{i_m}{d_c}$.

It is evident that in the situation the milk increased 1%, the cream decreased 36%, so that the ratio i_m/d_c is 1/36 which corresponds to our previous solution. We have here a very beautiful, useful formula which says that the ratio of the cream to the milk is inversely proportional to the changes in the per cent of butter fat content. Any dairyman could use this formula with the ease and accuracy with which he could compute the area of a circle, yet we, as his teachers of mathematics, although we did provide him with the formula for the area of a circle, have not provided him with the analysis nor the relationships which give him this powerful formula.

I have tried to point out that while answers to problems may sometimes be of great importance, it frequently occurs that an answer to a problem is of little consequence, but that a general answer to a problem may be of great consequence and very fundamental. Furthermore, when a formula is discovered which solves a problem, relationships are often revealed through the study of that formula which we probably never dreamed existed. As the means to be used for this analysis and the expression of the relationships involved in any quantitative situation, mathematics provides the formula, the graph, the table, and powerful techniques for revealing those relationships precisely, clearly, and without error. Through mathematics such as this, we can defend America by making its citizens intelligent in the analysis, the understanding, and the control of quantitative relationships.

An Experimental Comparison of Algebraic Reading Practice and the Solving of Additional Verbal Problems in Tenth Grade Algebra

By David H. Russell
The University of British Columbia, Vancouver, Canada

and F. Melville Holmes
Bedford Road Collegiate, Saskatoon, Canada

I. The Problem

The solution of verbal problems in algebra has always presented difficulties to pupils and teachers. Orlie and Hendershot conclude that "the weakest part of classroom instruction is the teaching of verbal problems" (8:147). Haertter says, "High School pupils who study algebra have greater difficulty in the solution of verbal problems than with the study of any other topic of that subject" (6:166). In a questionnaire submitted by one of the present writers, fifty-three teachers of algebra in the province of Saskatchewan ranked the solution of verbal problems as the most difficult topic in the algebra they taught.

Most writers agree that inadequate reading ability is an important factor in this poor performance in the solution of verbal problems. For example, Breslich states "The first training they should receive is in reading problems" (1:192). Other writers have gone a step further by attempting to show that reading practice favorably influences achievement in the solution of verbal problems. Most of these latter studies have shown somewhat superior achievement by groups that had reading practice *in addition to* their regular work in algebra. The present study differs from previous investigations in that it eliminates the factor of extra time spent in some type of algebra practice. Its purpose is to compare the effects of practice in various types of reading in algebra with the effects of spending the same amount of time actually solving verbal problems.

The purposes of the present study may be summarized as follows:

1. To compare the effects of more practice in solving problems with the effects of instruction in:
   a. Reading to interpret a problem
   b. Reading to recognize algebraic symbolism:
      (1) Conventions
      (2) Translating English into algebra
      (3) Translating algebra into English
   c. Algebraic vocabulary
   d. General mathematical reading
2. To find the improvement in (a) general reading ability and (b) algebraic reading ability as a result of the instruction given.
3. To find out which sex group benefits more from the reading instruction in regard to both algebra ability and reading ability.

The main purpose, then, is to help determine whether it is better to spend all available time of algebra class periods in solving verbal problems, or to spend part of that time on instruction in reading, and the remaining time on solving problems.

II. Related Studies

Reed, in his recent text, contends that "a search through the literature on the subject reveals much advice on how to solve problems, but little or no proof that the advice is good. This phase of secondary school mathematics has so far received little attention in the way of scientific investigation" (9:476). However, several attempts have been made to investigate the relationship between reading ability and the ability to solve problems in arithmetic and algebra. A few of these dealing with algebra will be discussed.

Hawkins attempted to determine the effectiveness of (1) practice in translating English expressions into algebra, and (2) analyzing problems, upon success in problem solving in ninth grade algebra (7). He found no difference in algebraic reading ability between the two experimental groups and the control group although no initial test was given in the ability to read. The gain in algebraic ability favoring the group which received the instruction in reading was small and was not shown to be reliable. Although Hawkins does not explain explicitly, apparently the reading instruction given to the experimental group was in addition to the regular work in algebra.

Stright found that specific training in reading and efficient methods of study improve the ability of students to do algebra (10). In this study no initial test was given in algebra, which makes it difficult to obtain a true picture of the relative progress of the two groups. It is again noted that the reading instruction is an additional feature for one group.

In another study, Clark claims that remedial instruction in reading during the regular algebra period "resulted in improved reading, which in turn effected a higher coefficient of correlation with algebraic achievement" (3:67). However, the experimental factor (the reading tests) is not described. Also, the time devoted to reading during the two weekly periods of remedial instruction, and the total time for algebra in both groups is not stated. The results obtained favoring the remedial group are not statistically significant. Other studies by Buckingham (2), Edwards (4), and Georges (5), give uncertain values to vocabulary as a factor in solving verbal problems.

Still other studies which claim values from instruction in reading likewise may be shown to be neither very reliable nor very conclusive. While it would seem that a definite relation should exist between a student's ability to read with comprehension and his skill in solving verbal problems, few studies, if any, have demonstrated unmistakably that the reading practice given during the experiment has had a reliably favorable effect upon achievement in solving verbal problems. To throw further light on the problem, the effects of practice in four main types of reading in algebra are compared in this study to the effects of direct practice in solving more problems.

III. Procedures

This study was carried out in four of the five grade ten classes of Bedford Road Collegiate, Saskatoon, Saskatchewan. All classes were taught by the same teacher for twenty-four lesson periods of thirty-five minutes each during January and February, 1940. The materials of instruction included the regular work on the solution of verbal problems using simultaneous linear equations, as found in the prescribed text for the province of Saskatchewan, supplemented by other problems of equal difficulty and involving similar procedures.

The experimental and control groups were matched on the results of four tests: (1) Mental ages as determined by the results on the Laycock Test of Mental Ability. (2) An Algebra Problems Test constructed by the authors. (3) Reading scores on the Thorndike-McCall Reading Scale. (4) An algebra Reading Test, consisting of items rather similar to those of the daily exercises in reading.

The results of the tests used for grouping are contained in Table I. The number of cases involved was lowered to 99 by withdrawals and by omitting pupils who missed two or more class periods during the period of the experiment.

The daily lessons for both groups were for thirty-five minutes. In the control group the entire period was used for the solution of verbal problems. No changes in the usual classroom procedures were made. In the experimental group the first fifteen minutes was spent using a reading exercise on a mimeographed sheet. The

TABLE I

A Comparison of the Mental Ages and of the Scores on Initial Tests of Two Groups of Tenth Grade Pupils

	Cases	Mental Age (years)		Algebraic Problems		Algebraic Reading		Thorndike-McCall Reading Scale (Form I)	
		Mean	S.D.	Mean	S.D.	Mean	S.D.	Mean	S.D.
Experimental Group	49	16.31	.65	4.80	2.16	29.49	4.46	29.61	2.08
Control Group	50	15.97	1.50	4.72	2.40	27.86	5.00	29.16	2.17
Difference		.34		.08		1.63		.45	

reading and answering of the material consumed approximately eight minutes, while the correction and discussion of the exercise occupied the other seven minutes. The remaining twenty minutes of each period was spent in doing the same problems as used in the Control Group insofar as time permitted. No work was allowed outside the class period in either group. Problems not found in the text were written on the blackboard.

Since the results of this experiment can be interpreted only in the light of the types of algebra reading exercises used, two of the exercises in reading to understand algebraic symbolism and to interpret a problem are given as samples below:

Reading Exercise 6

Many of the mistakes made in solving algebraic problems are due to the fact that symbols and words commonly used in problems are not understood, or are not properly interpreted in the problem. In each of the following problems the answer(s) to the question is to be underlined. Each question involves some word or symbol which you should know in order to read problems successfully.

1. In the following list of algebraic terms underline *two* which are *like* terms and circle *two* which are *unlike* terms.

   $3ab^2,\quad 3ab,\quad 7a,\quad 5ab,\quad 7ab^2$

2. Which of the following expressions is the "quotient of three times a number and the product of two other numbers"?

   $3a+bc,\quad 3a/bc,\quad 3a(bc)$

3. Which of the following terms correctly represents "four times the square of a certain number"?

   $(4a)^2,\quad 4/a^2,\quad 4a^24^2a$

4. "The cube of the sum of a and b" is:

   $(a+b)^3,\quad a^3+b^3,\quad (ab)^3,\quad (a/b)^3$

5. Which correctly represents "a is ten less than twice b"?

   $a-2b=10,\quad 2a=10-b,\quad a=2b-10$

6. The distance a boy rides a bicycle in x hours if he can ride y miles an hour is:

   $xy,\quad x/y,\quad y/x$

7. The cost of two books if ten books cost "m" dollars is represented by:

   $2(10m),\quad 2(m/10),\quad 10m/2,\quad 2(10)/m$

8. How many oranges can a lady buy for "a" cents if a dozen oranges can be bought for "c" cents?

   $a/12c,\quad 12a/c,\quad ac/12,\quad 12c/a$

9. Which expression represents "the square root of a number increased by the square of twice the number"?

   $X^2+(2x)^2,\quad \sqrt{x}+2x^2,\quad \sqrt{x}+(2x)^2,\quad \sqrt{x}-2x^2$

10. The sum of the coefficients of the terms of the expression $3x^3+7x^2+10x$ is equal to:

    6, 20, 25, 5.

Reading Exercise 22

Following each of the given problems are four answers to the problem, *one* of which is correct. In each case *circle* the answer which you would estimate to be *most correct. Carefully* study the problem before making your estimate but *do not solve* the problem. Simply mark the answer which seems most reasonable.

A. How many years must \$850 be invested at 5% simple interest in order to yield \$255?
   (1) 20 years (2) 5 years (3) 2 years (4) 12 years.

B. Find two consecutive even numbers whose squares differ by 52.
(1) 5^2, 54 (2) 5, 8 (3) 100, 102 (4) 16, 14
C. A man earned $78 income annually from investing $1500 in two firms, one investment earning 5% and the other 6%. How much has he in each investment?
(1) $800, $700 (2) $1250, $250 (3) $1500, $100 (4) $1000, $500.
D. A collection of nickels and dimes containing 46 coins amounted to $2.90. How many dimes and nickels are there?
(1) 45 dimes, 1 nickel (2) 4 dimes, 42 nickels (3) 35 dimes, 11 nickels (4) 20 dimes, 33 nickels.
E. A gentleman has two automobiles and a motorcycle worth $2400 in all. The value of the better car and motorcycle is three times the value of the cheaper car; and the value of the two automobiles is seven times the value of the motorcycle. What is the value of each?
(1) 1500, 600, 300 (2) 800, 800, 800 (3) 1000, 600, 600 (4) 1200, 800, 400.
F. If Robert can hoe half of the family garden in 10 hours and Chester can hoe all of it in 15 hours, how long will it take both boys to do it together?
(1) $1\frac{1}{2}$ hours (2) 19 hours (3) 8 hours (4) 25 hours.

The procedure used in the actual instruction in both groups was one of individual practice as much as possible. Considerable freedom of discussion of difficulties was allowed between students, and common difficulties were presented for class discussion. The various types of problems were introduced with explanation of general methods of attack. The types of problems considered might be classed as (1) business (2) number and digit (3) motion (4) "mixture" (5) mensuration (6) age (7) "work," and (8) miscellaneous.

IV. Results

The results of the initial and final tests are contained in Table II. In this table, the initial and final Algebra Problems Test (in items 1 and 2) are the same and consist of sixteen representative problems of the above types.

Table II reveals that the group which practised reading algebraic materials gained reliably more in such reading than the group which spent an equal time solving more problems of the same kind. This is an expected result since the content of the tests and the exercises was similar. It also reveals that there is no reliable difference between the effects of reading practice and solving more problems, as used in this study, on ability to solve verbal problems. The results of the first three items indicate an unreliably small, but consistent, difference in favor of the group which did more problems. This was as true of complete solutions as of deriving equations only, and was true for an immediate test of problem solving and one given six weeks later. The table further shows that on the Thorndike-McCall Reading Scale (a test largely of paragraph comprehension) the pupils of both groups showed a very slight loss between forms 1 and 2 of the test. This loss is slightly greater for the reading-practice group but is small in that the Thorndike-McCall Score has a Probable Error of 1.5.

Table II

A Comparison of the Mean Gains of the Experimental (Reading Exercises) and Control (Solving More Problems) Groups in Tenth Grade Algebra

Test	Maximum Score on Test	Control M Gain	Experimental M Gain	Difference C−E.	S.D.D.	C.R.
Algebra Problems—Correct Answers	16	3.28	2.73	.55	.378	1.48
Algebra Problems—Correct Equations only	16	3.40	2.98	.42	.355	1.18
Algebra Reading	45	3.74	6.14	−2.40	.691	3.47
Thorndike McCall Reading Scale	35	−1.93	−2.38	.45	.467	.96
Algebra Problems—Delayed Test	10	3.64*	3.25*	.39	.39	1.00

* Actual test scores only

A comparison of boys' and girls' scores also reveals no reliable differences between them in either the control or the experimental groups. However, the girls gained slightly more than the boys in the control group (a difference of half a question out of sixteen questions) and the boys seem to gain slightly more than the girls in the experimental group (a difference of about one question in a test of sixteen questions). In no cases are the results reliable, the size of the separate groups averaging only twenty-five.

V. Interpretation

The results of the study are not conclusive in regard to most of the comparisons made. The only completely reliable result indicates that instruction in the reading of algebra is valuable in itself. There is some indication that boys profit more than girls in this respect. This may be due to the fact that boys on the average start with less linguistic ability than girls and specific instruction tends to lessen the gap.

All results in this study must be interpreted in light of the time allotments used and the type of exercises practised. Apparently it is slightly better to teach verbal problems in an accepted manner for the entire thirty-five minutes than to teach them in the situation where the time allotment is fifteen minutes for instruction in algebraic reading and twenty minutes for instruction in the problems. This may not be true when the reading practice is ten minutes out of forty minutes or when more than six practice lessons in any one of four types of reading can be given. However, since the results claimed in other studies for the value of reading instruction have been found to be largely unreliable or insignificant, is it possible that these claims for the benefit of reading instruction are not justified by actual experimental evidence when compared with other types of instruction? Such would appear to be the case in the solution of verbal problems, when the reading exercises are of the type used in this study and in the investigations quoted above. This implies that some more important factor or factors, other than reading ability, exist which influence the ability to solve problems. The determining of these factors would be a worth while outcome of a series of coöperative studies in the field.

It may be noted that the algebraic progress of both groups in the present study was satisfactory. However, the interest of the students, and their enjoyment of the work, seemed greater under the experimental conditions. Since the actual differences in progress were small and unreliable statistically, it may be inferred that the total progress of one group was as great as that of the other. This study, like preceding ones, does not demonstrate that practice in reading algebraic materials necessarily affects favorably the ability to solve verbal problems in algebra. Of course, in the present study, the factor of time spent on algebraic activities was controlled and in actual, everyday practice it is often possible for the teacher to give help in how to read verbal problems in study periods or at home. However, this study underlines two present-day needs. Further investigation of factors other than reading ability that affect skill in problem solving is needed; and more exact knowledge of the nature of the reading abilities required in solving verbal problems is needed before reading exercises can be prepared which will have a definite influence upon achievement in solving verbal problems in algebra.

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Factoring Has a Meaning

By THOMAS H. MURTAUGH, B.Sc., C.E.
School of Education, Fordham University, New York City

WHY FACTORING?

ALL MATHEMATICS teachers on the secondary level have at some time or other experienced difficulty in making abstract but necessary mathematical principles purposeful and vital to students. The average high school student often asks, "What's the use of learning such meaningless operations as $(a^2-b^2)=(a+b)(a-b)$?" The student who gives expression to this thought is not "de facto" the "ne'er-do-well." He is the student who has factored "c squares minus d squares" until he has felt like "the difference of two squares." He joins other students who, ordinarily docile, have risen in sheer consternation and now demand to know the use, the purpose, the function—the part such rules as $a^2-b^2=(a+b)(a-b)$ play in their lives.

A teacher who passes over such rules as $(a+b)^2=a^2+2ab+b^2$ and $(a-b)^2=a^2-2ab+b^2$ without showing that these rules actually represent areas and are, if applied, short-cuts in the operation of multiplication, IS NEGLECTING TO IMPART ESSENTIAL INFORMATION TO HIS STUDENTS.

Certainly, a boy or a girl who has attained the high school level must have acquired some knowledge of simple areas and surely must have learned that in business and professional life these rules and facts may at some time or other be applied. In simpler words a teacher will not talk over his students' "heads" if these facts are presented clearly, concisely, and logically.

The Difference of Two Squares Specifically

Does $(a+b)(a-b)=a^2-b^2$ mean only that the product of the sum of two numbers by the difference of the same two numbers equals the difference of their squares? And vice versa, does $a^2-b^2=(a+b)(a-b)$ mean only that the difference of the squares of two numbers equals the product of the sum of their roots by the difference of their roots? Has this rule any physical significance? Can this rule be visualized? Are there any practical applications which may be made of this rule? Can the teaching of such a rule be made more worth while, interesting, and effective by the demonstration of these physical, visual, and practical applications?

PHYSICAL APPEARANCE OF
$a^2-b^2=(a+b)(a-b)$

What does (a^2-b^2) look like? Figure 1 shows clearly at a glance the appearance of a^2-b^2.

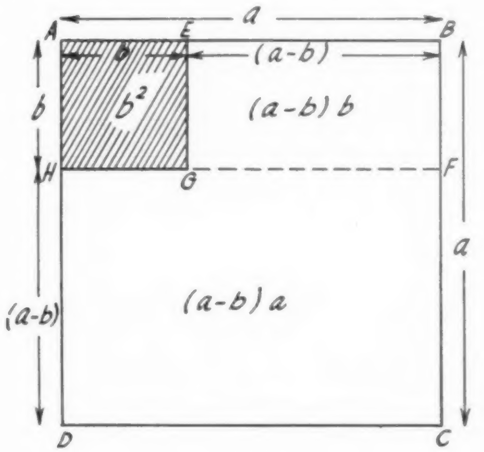

FIG. 1

"a^2" is an actual square; namely, $ABCD$, the sides of which AB, BC, CD, and DA are each "a" units long.

"b^2" is likewise an actual square; namely, $AEGH$, the sides of which AE, EG, GH, and HA are each "b" units long.

Therefore a^2-b^2 is that position of the diagram which is unshaded and is composed of two rectangles; namely, $EBFG$ and $HFCD$.

In the rectangle $EBFG$, EB is $(a-b)$ and BF is "b." The area is, therefore, $(a-b)b$.

In the rectangle $HFCD$, HF is "a" and FC is $(a-b)$. The area is, therefore, $(a-b)a$.

The sum of the two areas is $(a-b)b$ plus $(a-b)a$. Since $(a-b)$ is common then the addition becomes $(a-b)(a+b)$.

Thus visually and physically $a^2-b^2 = (a+b)(a-b)$.

Practical Applications

$$(a+b)(a-b)=a^2-b^2$$

What are the products of 52 by 48, 94 by 86, 88 by 72, 216 by 184?

Short-Cut in Multiplication

The foregoing products can be found mentally by the application of the rule of the difference of two squares. Note that in each product problem one number is the same numerical value above a common order as the other number is numerically below the common order.

In 52×48, the common order is "50" and 52 is $(50+2)$ while 48 is $(50-2)$. Therefore,

$$\begin{aligned}52\times48&=(50+2)(50-2)\\&=2500-4\\&=2496.\end{aligned}$$

In 94×86, the common order is "90" and 94 is $(90+4)$ while 86 is $(90-4)$. Therefore,

$$\begin{aligned}94\times86&=(90+4)(90-4)\\&=8100-16\\&=8084.\end{aligned}$$

In 88×72, the common order is "80" and 88 is $(80+8)$ while 72 is $(80-8)$. Therefore,

$$\begin{aligned}88\times72&=(80+8)(80-8)\\&=6400-64\\&=6336.\end{aligned}$$

In 216×184, the common order is "200" and 216 is $(200+16)$ while 184 is $(200-16)$. Therefore,

$$\begin{aligned}216\times184&=(200+16)(200-16)\\&=40000-256\\&=39744.\end{aligned}$$

All these results have been obtained mentally by the application of the rule for the difference of two squares.

$$a^2-b^2=(a+b)(a-b)$$

A second application arises in finding the area of a walk of uniform width about a square, and in finding the area of a circular ring.

A Short-Cut in Certain Area Problems

Problem 1. What is the area of a 4 ft. walk around a grass plot which is 20 ft. by 20 ft.?

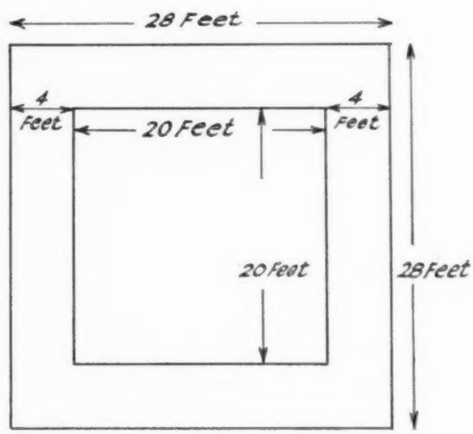

Fig. 2

Referring to Figure 2 the area of the walk is the area of larger square, 28 ft. by 28 ft., minus the area of the smaller square, 20 ft. by 20 ft. This, according to the rule is

$$\begin{aligned}28^2-20^2&=(28+20)(28-20)\\&=48\times8\\&=384 \text{ square feet.}\end{aligned}$$

The preceding approach is very quick and simple, reducing the solution to mere mental operations.

Problem 2. What is the area of a circu-

lar ring the inner radius of which is 12 feet and the outer radius of which is 15 ft.?

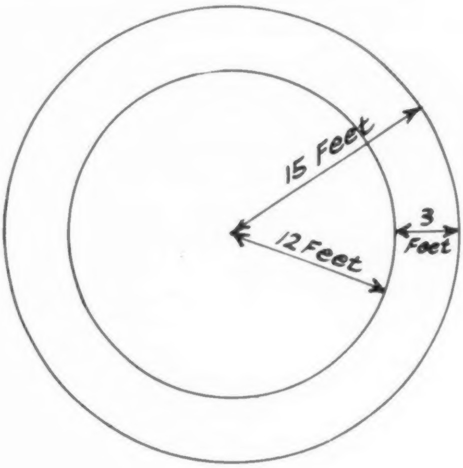

FIG. 3

Referring to Figure 3 the area of the ring is the area of the larger circle, $\pi \times 15 \times 15$, minus the area of the smaller circle, $\pi \times 12 \times 12$. This according to the rule is

$$\pi(15^2-12^2)=\pi(15+12)(15-12)$$
$$=\pi \times 27 \times 3$$
$$=254.34 \text{ square feet.}$$

Application of

$$a^2-b^2=(a+b)(a-b)$$

to the Pythagorean Theorem

A third application arises in the use of the Pythagorean Theorem, given the hypotenuse and one leg to find the other leg.

In the triangle ACB, Figure 4, the hypotenuse is "c," and each leg is "a" and "b" respectively.

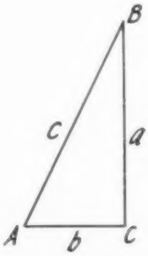

FIG. 4

By the rule of Pythagoras $c^2-a^2=b^2$ *and* $c^2-b^2=a^2$, that is $(c+a)(c-a)=b^2$ *and* $(c+b)(c-b)=a^2$.

Of course, an additional operation is necessary in this case; namely, the square root of "b^2" or "a^2" to give "b" or "a" respectively.

PROBLEM 3. A thirty-five foot ladder must be placed against a building to reach a height of 30 feet from the ground. How far from the building should the ladder be placed?

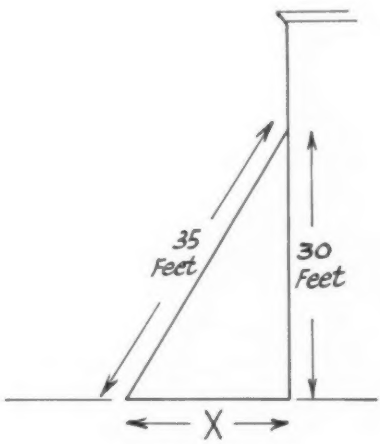

FIG. 5

$$x^2=35^2-30^2$$
$$x^2=(35+30)(35-30)$$
$$=65 \times 5$$
$$=325$$
$$x=\sqrt{325}$$
$$=18.02 \text{ feet}$$

The foregoing solution is entirely mental save for the square root which can be approximated in actual practice.

Use of

$$a^2-b^2=(a+b)(a-b)$$

Saves Human Lives

PROBLEM 4. An 85-foot aerial ladder must reach a point on a building which is 79 feet from the ground. How far from the building should the ladder be placed?

An 85-foot aerial ladder, employed by Fire Departments of large cities, is 85 feet when extended to its full length vertically,

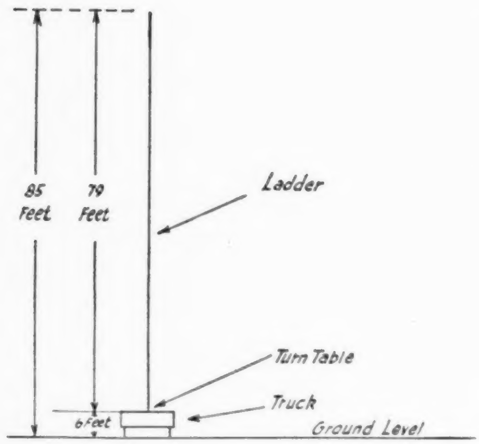

Fig. 6

Figure 6. Note that the truck is 6 feet from the ground to the turn table of the ladder while the ladder is actually only 79 feet. Hence in this problem the picture is shown in Figure 7.

Here $x^2 = 79^2 - 73^2$
$= (79+73)(79-73)$
$= 152 \times 6$
$= 912.$
$x = \sqrt{912}$
$= 30.19$ feet.

In these latter cases a man who has to think quickly where human life is in jeopardy can certainly arrive at his result quickly by applying the rule that the difference of two squares is merely the sum of the roots times the difference of the roots.

These are by no means all the applications which can be made of this one rule of factoring. There are many, many others. The teacher, however, who makes an attempt to show his or her students how to apply such rules, and where such rules may fit into life and occupational situations later on IS DOING SOMETHING WORTHWHILE FOR THE STUDENT AND THE MATH COURSE.

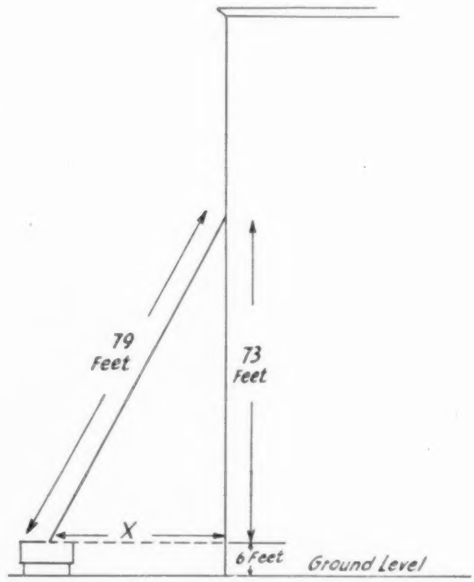

Fig. 7

A Dream

(After Reading "Imagination and Mathematics.")

A Lily on a windless pond
Steps a crystal door beyond,
Perfect line for perfect line,
Inversely beauty to define—
Till sudden breezes ruffle
The surface, lovely lines to shuffle,
But alack! in true proportion,
In a twisted, wry distortion.

Aloft from fields Elysian
Stoop once more to earthly vision,
Plato, Euler, each one sees
Their various twain geometries
Illustrate. Frowns the latter,
Whisp'ring, "Doth it seem to shatter,
Master, of the pure ideal
The image—which one is the real?"

But Plato, nowise troubled, thus
Makes answer, "Sweet Discipulus,
Fret not thyself with query's tedium,
Surely we now know mortal medium,
Infinitely manifold,
Must make till every star grow cold
Each man how strangely view aslant
The divine invariant!"

Anne Young

A Prediction of Pupil Success in College Algebra[1]

By WINSTON M. SCOTT *and* JOHN P. GILL
University, Alabama

1. NO PERSON will propose that he can predict the mark of a particular pupil. We do claim, however, that, within certain limits of error, the marks of a group of students can be predicted.

This work was begun in an attempt to find out why such a large percentage of freshmen fail college algebra. Our sample is taken from the records of 1103 algebra students of six different years. We realize, of course, that the mark of a student in college algebra depends upon a great many variates, but we feel that two are of particular importance. These two are: the number of units of algebra taken in high school and the number of years intervening between the time the last unit of algebra was taken in high school and the date of entry into college algebra.

We have not been particularly interested in other college mathematics marks for we contend that ability evidenced in college algebra will at least carry one through freshman mathematics courses.

2. We shall denote by x the number of units of high school algebra, by z the number of years intervening between the last year of high school algebra and entrance into college, and by y the grade point ratio[2] made in college algebra.

Although we know that the relation among the variables may not be a linear one, we shall use the linear relation

$$y = C_0 + C_1 x + C_2 z, \tag{1}$$

where C_0, C_1, and C_2 are constants which we shall determine from our sample, as an approximate relation.

From (1) we can, for any given x and z, obtain a predicted value of y dependent upon C_0, C_1, C_2, x, and z.

For each particular student of our sample of 1103, the error in predicting his grade in this manner could be written:

$$\epsilon = {}_0y - (C_0 + C_1 x + C_2 z)$$

where ${}_0y$ is the actual observed grade in college algebra. The general equation of the 1103 of this sort, then, becomes

$$\epsilon_i = {}_0y_i - (C_0 + C_1 x_i + C_2 z_i).$$

Minimizing the sum of the errors squared, we have from

$$\begin{aligned}\sum_{i=1}^{1103} \epsilon_i^2 = {} & \sum_{i=1}^{1103} {}_0y_i^2 + NC_0^2 + C_1^2 \sum_{i=1}^{1103} x_i^2 \\ & + C_2^2 \sum_{i=1}^{1103} z_i^2 - 2C_0 \sum_{i=1}^{1103} {}_0y_i \\ & - 2C_1 \sum_{i=1}^{1103} x_i\, {}_0y_i - 2C_2 \sum_{i=1}^{1103} {}_0y_i z_i \\ & + 2C_0C_1 \sum_{i=1}^{1103} x_i + 2C_0C_2 \sum_{i=1}^{1103} z_i \\ & + 2C_1C_2 \sum_{i=1}^{1103} x_i z_i,\end{aligned} \tag{2}$$

by taking the partial derivative with respect to C_1 set equal to zero, that

$$\begin{aligned} & N\sum x_i\, {}_0y_i - \sum x_i \sum {}_0y_i \\ & = C_1[N\sum x_i^2 - (\sum x_i)^2] \\ & \quad + C_2[N\sum x_i z_i - \sum x_i \sum z_i].\end{aligned} \tag{3}$$

In the same manner, the partial derivative with respect to C_2 gives, when set equal to zero, that

$$\begin{aligned} & N\sum z_i\, {}_0y_i - \sum z_i \sum {}_0y_i \\ & = C_1[N\sum x_i z_i - \sum x_i \sum z_i] \\ & \quad + C_2[N\sum z_i^2 - (\sum z_i)^2],\end{aligned} \tag{4}$$

[1] This study has been based upon students regularly enrolled in the Business Administration School of the University of Alabama who completed at least one semester of work.

[2] The grade point ratio at the University of Alabama is determined on the basis of 3 quality points for each semester hour of A's, 2 for each hour of B's, and 1 for each hour of C's.

and the partial derivative with respect to C_0, equated to zero, gives

$$(5)\quad C_0=\frac{\sum {}_0y_i-C_1\sum x_i-C_2\sum z_i}{N},$$

where $N=1103$.

From the data of the sample the values of x, y, and z were available. We then have (3), (4), and (5) as

$$(3')\quad 402{,}012=314{,}268C_1-455{,}836C_2,$$

$$(4')\quad -768{,}309=-455{,}836C_1+2{,}790{,}565C_2,$$

$$(5')\quad C_0=\frac{317-1926C_1-2201C_2}{1103}.$$

These give

$$C_1=1.153045,$$

$$C_2=-0.086975,$$

$$C_0=-1.552433,$$

which reduces (1) to

$$y=-1.552433+1.153045x-0.086975z.$$

Again, it should be noted that this equation of prediction (if linearity of the relation is not too big an assumption) is not to be applied to particular cases, but is used only as a means of predicting the percentage (within certain limits of error) of students who will have any given grade point ratio. Some large companies have recently adopted the plan of giving prospective employes examinations and using the results of these examinations in the personnel department. This, of course, does not necessarily mean that in any particular case the prospective employe should or should not be hired, but, applied to the large groups hired, it has been found that the results aid in selection, and get the largest percentage of the most capable men.

3. By computing the value of the standard deviation of the errors it is possible, for any large group of students, to predict the number who will have any given grade point ratio (assuming that the number of units of high school algebra and the number of years intervening between the time at which the last of these units was studied and enrollment in college algebra are both known).

The value of the standard deviation of the errors is given by

$$\sigma_\epsilon=\frac{1}{N}\sqrt{[N\sum y^2-(\sum y)^2]-C_1[N\sum xy-\sum x\sum y]-C_2[N\sum yz-\sum y\sum z]}$$

or, for our sample,

$$\sigma_\epsilon=1.539465.$$

To find the percentage of pupils who will have l as a grade point ratio, we let

$$t=\frac{l-(C_0+C_1x+C_2z)}{\sigma_\epsilon}.$$

Pearson's tables of areas, then, enable us to compute the probability that l will be the grade point ratio.

As an example, suppose a group of students had two units of high school algebra, the last one of which was taken two years before entering college. To find the percentage of this group who should make A's (or have a quality point ratio of 3), we proceed as follows:

$$t=\frac{3-[-1.552433+1.153045(2)-0.086975(2)]}{1.539465},\quad t=1.57216.$$

From Pearson's Type III Curve Table we find the area bounded by the curve below the ordinate $t=1.57216$ to be 0.94186. Therefore, of a group of students having two units of high school algebra, the last one of which was studied two years prior

to entrance into college, we have that 94.2% should make less than a "3" grade-point-ratio, or, that approximately 5.8% should make "A" in college algebra.

Following this procedure, we have, on the basis of our assumption that the equation of prediction is linear, from our sample of 1103 pupils,[3] the following table.

Mark	Group I $x=2$ $z=2$	Group II $x=2$ $z=0$	Group III $x=1$ $z=3$	Group IV $x=2$ $z=1$
"A"	.058	.073	.009	.065
"B"	.120	.136	.033	.134
"C"	.215	.228	.099	.215
"D"	.254	.215	.193	.253
"F"	.353	.348	.666	.333

4. From this it seems evident that the important thing is the number of units of high school algebra which the student has had. Also, it is seen that the time intervening between the last year of high school algebra and college algebra has a comparatively small effect on the college algebra mark. In other words, we think that most of our failures can be directly attributed to lack of high school algebra.

There are a great many other factors entering into any determination of success in college algebra. Among these the high school attended and the marks attained in high school algebra should certainly affect the college algebra mark. A study with these two as variates added to the two used here could be made by following the same procedure.

One might also have assumed (1) to be exponential, or, it might have been taken to be quadratic, and with either of these as the basic form of the equation of prediction, a study based on the data used here could have been made.

[3] It might be of interest to note that of our sample of 1103, 222 fell in Group I, 135 fell in Group II, 203 in Group III, and 289 in Group IV of the table below. This is 76.9% of the total sample.

Rob Roy's Problem and Answer

THE PROBLEM

A column of infantry one mile long is marching foward at a constant rate of speed. An officer, mounted on horseback, starts at the rear of the column and rides forward at a uniform rate of speed which is faster than that of the infantry. He rides until he catches the head of the column and then instantly turns back and rides again to the rear of the column, reaching the rear at precisely the moment it passes the point at which the head started when the officer began his ride. How far did the officer ride? No time is allowed for his turning around to ride back.

THE SOLUTION

Let d = distance from point at which head of column started to point at which head is overtaken by officer

D = total distance travelled by the officer $=1+2d$

To reach the head of the column the officer rides a distance $1+d$ while the column marches a distance d. The officer then turns back and rides a distance d to the rear of the column, reaching it just as the rear marches to the starting point of the head. During this return ride the head of the column has marched a distance $1-d$. Since the speeds are uniform and the time the same for both column and officer these distances will be in proportion:

$$\frac{d}{d+1}=\frac{1-d}{d}$$
$$d^2=1-d^2$$
$$d^2=\tfrac{1}{2}$$
$$d=\sqrt{\tfrac{1}{2}}$$
$$D=1+2d=1+2\sqrt{\tfrac{1}{2}}=2.414 \text{ miles}$$

From *The Kalends*, Waverly Press, Baltimore, Md.

Teaching the Laws of Algebra, or Mathematics Is a Language

By Carl Denbow
Ohio University

In recent years the old definition-rule-drill method of teaching algebra has given way. The concise, deductive systems inspired by Euclid have been "kicked upstairs" to the college or post-graduate level.

None of these improvements, however, has aided the teaching of that basic part of algebra, the laws of algebra. In fact, the laws of algebra suffered from their resemblance to mechanical rules and were often discarded simultaneously. But this was throwing out the baby with his wash-water, as the proverb says. These laws, far from being dull, can actually be used to motivate the student by making the entire subject more interesting; they are, in fact, the only means for bringing meaning into the chaos which algebra appears too often to be. For it is no exaggeration to say that one's understanding of algebra, as distinguished from his manipulative skill, is proportional to his insight into the purpose and the meaning of the laws of algebra. The aim of this paper is to give a simple—and abridged—exposition of some of these laws, using what we may call the language method, and to suggest some applications to classroom teaching. First the language method is considered briefly.

Two aspects of algebra shall concern us; on the one hand the truths of algebra, on the other, the manner or language in which they are expressed. Here we use "language" in its general sense, "a means of communicating ideas." Thus, traffic lights and the signals of football referees are languages. And any language is abstract and arbitrary. No symbolism of the language of algebra is more abstract than the use of a green light to mean "Go." This green light symbol, though very familiar, has no logical connections with the verb "Go," but has been arbitrarily chosen.

This arbitrariness of language must be grasped before algebra can be clearly understood, but it is not usually seen by students without some help. They feel that somehow the letters "d-o-g" *must*, inevitably, in the very nature of things, mean a member of the canine family. Many people, in fact, seem to believe that words have certain inherent meanings, apart from choice or custom. Hence it is not surprising that it is frequently not the logic nor the mechanical manipulations of algebra which cause most trouble for the learner; it is the language aspects. The unaided student must grope his way, and often thinks he fails to grasp the logic of algebra, when there is nothing to learn but the meaning of a new "word." Two examples may make this clear.

When a student says, or vaguely feels, that he doesn't understand *why* $\sqrt[3]{x}\cdot\sqrt[3]{x}\cdot\sqrt[3]{x}=x$, what he needs to be told is that $\sqrt[3]{x}$ is merely a name, a word used for a number which upon multiplication by itself twice gives the number x. That is to say, it is a *fact* that there is a number such that, multiplied by itself twice, the result is x, but it is man's own *choice* to give that number the *name*, "$\sqrt[3]{x}$." There is no magic in this name; it does not even tell us how to find the number in question, though mathematicians have learned methods of computing it.

Some teachers object to such an explanation, because it does not fall into the logical deductive scheme; or they feel that it is too simple and that no student needs to be told such obvious facts. We shall not discuss these objections here, but give a second example.

How many students think of algebra as "the x, y, z business!" Or on a higher scale, that algebra uses letters for numbers! This seems very strange to them, for

how can the letter x be a number? And if x equals 5, why not use 5? Why use x? Such questions may seem childish to the trained mathematician, but answering them, or teaching so that they never arise, is one of the teacher's chief tasks. In the sequel we shall try to show: (1) that one can teach algebra as a language; (2) how to make clear to one's students that new words may be chosen arbitrarily; (3) that algebra needs new words; and (4) that x is a word chosen to mean "any number" at times and "some number" at other times. x is not a number, but a word standing for a number. Algebra is not the study of x and y, but uses these words in studying numbers. This distinction need never be explained to the student of history; he knows, without being told, that history is not the study of the words, "Washington," "English Army," "Delaware," etc., but of the men and objects named by those words. Yet he may take algebra for a year and think it deals mainly with the *words* "x" and "y." It is possible to prevent such errors; to teach the *meaning* of words, equations, etc., in algebra.

The importance of language in arithmetic can easily be shown students, thus paving the way for algebra while clarifying arithmetic. For example, addition and multiplication take much longer if we write out numbers in English, "Three hundred forty-five plus five hundred thirty-seven." In Roman numerals also, "XXXII times XLIX" is a difficult problem. Now XXXII is the *same number* as 32, but written in a different language. The number language we use makes mechanical manipulation easy. Any 10-year-old boy can *mechanically*—using only memorized rules and no thinking—make computations that taxed the powers of great thinkers 2000 years ago.

Further discussion of the language aspects of algebra leads us back to our main point. We believe that the chief difference between arithmetic and algebra is a basic difference in purpose, in point of view. Algebra seeks principles or truths holding for *all* numbers (and, of course, studies the applications of these principles to particular cases). This we shall attempt to show. We may first define arithmetic, roughly, and with actual teaching practice in mind, as the study of computation with integers, common and decimal fractions, and square roots, plus practical applications. ("Computation" here means the four fundamental operations plus the extraction of square roots; raising to powers, percentage problems, etc., are thus included.) After learning these basic procedures one tries to discover general truths for all numbers, just as (the teacher may say), a doctor tries to find a cure which works for all people.

This point of view not only clarifies algebra and gives meaning to the "Laws of Algebra," as seen below, but it also interests the students, for they like to be told that algebra is not primarily computation, but a series of discoveries in which they can share. We can tell them that any general truth about numbers is called a law of algebra and ask them to find some, the simpler the better. Some will say that any two numbers may be added; others that, of any two numbers, one is always the larger. Others will say that $5+3=3+5$, and so on *for all numbers*. At this point they can be led to see how complicated is the complete statement in English of this simple truth, and that a language of algebra is needed.

At this point the teacher may well say: "I shall give you another very interesting law of algebra. It is complicated to state, but easy to test, and useful in many ways. It is this: if any number whatever is squared and one then subtracted from the result, the answer obtained is the same as the number found as follows: take the original number and add one to it, then subtract one from the original number, and multiply these two numbers together." The writer has several times asked his students to copy this down, and then has watched the befuddled and skeptical looks change to eagerness as they see that $9^2-1=80$ and $(9-1)(9+1)=80$, and that

$6^2-1=35$ while $(6-1)(6+1)=35$, etc., until they become convinced that this complicated statement is true for all numbers. The teacher then continues: "Surely such a powerful and simple truth can be stated in simpler words. We can use the language of arithmetic and say: (any number)$^2-1=$(that number$-1)\times$(that number$+1$). (Note: the students can make this "translation" with a few hints: they all see, for example, that "the answer obtained is the same as . . . " can be written "$=$".) Can we not choose a new word for 'any number'? If manufacturers can invent such ugly words as 'gap-osis' and if someone can foist 'jitterbug' on an innocent world, surely we have the right to invent a word which will simplify algebra in the same way that the change from Roman to Arabic numbers simplified arithmetic. Clearly, one-letter words are the simplest, so we shall use 'x' to mean 'any number. . . . "

Next, after introducing "y" to mean "any number, perhaps different from x," one can give the example $4+7=7+4$, and nearly every student can then state in English the general truth, "Any number plus any other number is equal to the second number plus the first" and in algebra "$x+y=y+x$." Each student sees the advantage of algebraic language, and, what is more important, from the very beginning he learns that equations *convey information,* and that x and y are used not as letters but as words standing for numbers.

From this point on progress is more rapid. One or two examples of the kind, $4-4=0$, $8\frac{1}{2}-8\frac{1}{2}=0$, and each student can express, in English and in algebra the general truth. $x+y=y+x$, $xy=yx$, $x-x=0$, $1\cdot x=x$, $x/x=1$ (when $x\neq 0$), $x\cdot 0=0$ and $x+0=x$, and possibly $x(y+z)=xy+xz$, are developed. One should emphasize that these laws, though simple, are very important, for practically everything in algebra can be proved rigorously from them, as plane geometry is proved from its postulates. *No rules,* such as "both terms of a fraction may be multiplied by the same factor," *are needed* in algebra. For example, $3/3=1$, so that $1/2=1/2\cdot 3/3=3/6$ is seen by logic, not by rule. In a later paper the writer hopes to show in detail how the laws of algebra can supplant "rules."

The recent paragraphs have rushed us through the very basis of algebra with hardly a breathing spell, and with only a few remarks on the teaching of the ideas touched on. In a later paper the writer hopes also to consider the teaching problems in more detail. Let us notice here, however, that one can teach students to discover for themselves that $(x+y)^2=x^2+y^2$ is *not* a law of algebra merely by taking $x=2$, $y=1$, for example. This equation simply isn't true for *all* numbers, so in all parts of algebra where we are dealing with *general truths* this equation is false. It is of course true for the special cases where $x=0$ or $y=0$. Similarly the students learn empirically, for themselves, that $x+x\neq x^2$, that $1/x+1/x\neq 1/2x$, etc. For further contrast with the laws of algebra one can show that many facts are true only for certain numbers, for example $2+2=2\cdot 2$ but $3+3\neq 3\cdot 3$, and $3^2+4^2=5^2$ but $4^2+5^2\neq 6^2$.

In the outline above the writer is seen to oppose the trend by some teachers toward beginning algebra, not by study of laws of algebra, but with conditional equations, such as $2x+3=9$, or word problems leading to such equations. (In $2x+3=9$, it is obvious that x does not stand for *all* numbers but rather for *one* "unknown" number, namely 3; similar cases often occur in English.) The writer's reasons for preferring the law of algebra approach cannot be enumerated here, but it should be emphasized that the language method can be used with either approach. In fact, its uses are almost unlimited. The change from $\times$ to $\cdot$ for multiplication, and the omission of the $\cdot$ in xy, have close English analogies. Just as a change of punctuation may change the meaning of an English sentence, so $2\cdot 3+4$ is ambigu-

ous, equalling 10 or 14, until we *agree* to perform the multiplication first *or* to use the parenthesis and write $2\cdot(3+4)$, depending on the meaning we wish to convey.

Other examples could fill articles longer than this one. But the use of such analogies is not fully effective unless carried out systematically, along with some discussion of the arbitrariness of language. Students are often told that mathematics is "logical" and they carry the idea so far that they have a feeling of discomfort when told "We *agree* to do such and such" or "This means so and so by definition." The language method makes clear that logic comes in after, and only after, language agreements have been made. For example, after we *agree* that x^2 means $x\cdot x$ and x^3 means $x\cdot x\cdot x$, we can prove that $x^2\cdot x^3=x^5$.

In concluding, the writer wishes to disclaim any originality whatever for the basic ideas of the paper; only the applications are new. Every research worker who has introduced new words into the large vocabulary of higher mathematics knows that mathematics is a language. It is noteworthy that in nearly every field of modern life the analysis of language is of growing importance. We mention here only R. Carnap's "Logische Syntax der Sprâche"; and that fine popularization, L. Hogben's "Mathematics for the Million" which stresses the linguistic aspects of mathematics.

The writer feels strongly that the chief obstacle between algebra and the wider public favor and recognition it deserves is our failure to "get across" the meaning of algebra, and especially the laws of algebra. The use of x in solving word problems is clearer to student and public than the value of equations like $x^2-1=(x-1)(x+1)$, called "factoring," or like $1/x+1/y=(x+y)/xy$, called "adding fractions." Students who have no trouble understanding or solving $3x-12=x+2$ may never see the point to factoring. It is hard to overestimate the importance of teaching these as identities or laws of algebra *from the beginning,* so the student never thinks of them *except* as methods of stating truths about all numbers. In the writer's classes nearly all of the students after four weeks know, for example, that $4x+9=17$ means "4 times a *certain* number$+9=17$" while $3x+5x=8x$ means "3 times *any* number $+5$ times the same number$=8$ times the same number," and that the first equation can obviously only be used when we are working with the number $x=2$ while the second can be used freely, any time, without question. In short, the writer believes that algebra can be made understandable. We do not advocate returning to the "page after page of factoring and fractions" method, but teaching a little and teaching it more thoroughly.

The fourth Dimension of Super-Spatial
It bears the same relation to Space
That Space does to Flatness.
It may be said to be Continuity.
Or Speaking poetically
It is the Shadow
That Time casts on Eternity.

Learning the Harder Multiplication and Division Facts in a Program Emphasizing Understanding

By HERBERT F. SPITZER *and* MAXINE DUNFEE
University of Iowa Elementary School, Iowa City, Iowa

A SURVEY of the current literature in the field of arithmetic reveals that much attention is now being given to the development of understanding. There is almost universal agreement among writers that facts should be meaningful and that drill should only be used after understanding has been attained. The use of meaningful procedures is frequently suggested as the method whereby understanding is to be attained. Classroom teachers and others closely associated with children in schools, however, realize that the suggestion "use meaningful procedures" does not go very far in the way of supplying specific methods to be used in teaching. The present paper describes a fourth grade program, in which primary emphasis is put on understanding of facts and processes, and in which learning procedures designed for mastery of facts were given a secondary place in time of introduction and in importance.

The area of arithmetic with which this paper is concerned is the learning of the multiplication facts (and corresponding division facts) from 6×6 to 9×9, inclusive. In teaching these facts, use was made not only of commonly accepted procedures, but also of procedures intended primarily for promoting understanding. The seven steps listed below identify these latter procedures:

a. Confronting the children with comprehensible problem situations which illustrate the new facts and processes to be taught.
b. Permitting children to solve these problems by using known methods although longer and more cumbersome than those which would be used when the new facts are mastered. In other words, permitting children to solve problems illustrating new facts and processes before explanations of facts and processes are given.
c. Guiding children's thinking, in the discussion and presentation of their own solutions, toward the discovery or seeing of new facts and procedures.
d. Requiring children to prove the truth of facts, and directing these proving exercises to show the true nature of processes and the interrelated nature of different processes.
e. Using oral or mental work to avoid the mere learning of symbols.
f. Assembling facts to show relationships and using various solutions to show how facts and processes are related.
g. Letting children discover through experience the economy of time and effort resulting from knowing the best procedures.

While the actual use of most of these steps will be obvious, for ease of reading, at least one instance of the use of each step has been identified by means of a footnote.

As a first lesson, the children were given a group of problems, the economical solution of which necessitated the use of some of the facts. No comment was made by the teacher other than that the children were to try to solve the problems in any way which seemed possible. The following are a few of the problems used:

1. If there are six rows of desks in the fifth grade room, and six desks in each row, how many desks are there in the room?
2. If a child reads 6 pages of a book each day for a week, how many pages will he have read?

3. If red pencils cost 6¢ each how much will 9 such pencils cost?
4. Will there be enough apples for 50 people at out party, if eight children each bring six apples?

As soon as the assignment was made, every child started to work. This expression of self-confidence on the part of the children can probably be attributed to the fact that they were not told that they were learning something new, and to the fact that the problems were within their experiences. Since the children had free rein in so far as the solving of these problems was concerned, many ingenious and unusual (although quite sensible) solutions were offered.[1] The teacher made available small objects such as toothpicks, paper clips, or sticks, which some children used in finding the answers. The use of these objects, of course, took much time. Some took an entire period to find one answer; others finished quickly. This did not disturb the teacher who let the slow ones plod along although she challenged or directed the thinking of some who were on the wrong track or had stopped with a wrong solution. For example, she said to a child, "Draw a picture of a schoolroom that has six rows of desks and six desks in each row. Now, how could you find out how many desks there are?", etc.[2] Some of those who showed that they knew just what they were doing were asked to make up simple story problems of their own which might be given to the class later. This was a favorite activity, for the teacher often used the problems at some other time. These pupils were also required to provide the answers to their own problems. This extra work was part of the plan to allow every child to discover for himself several of the new facts before time was taken for class discussion.

At the close of the work period, a presentation of the children's answers and methods of solution was made. There were several favorite ways to solve problems. Many used objects or marks to represent the numbers and then counted to find the total. For example, in the first problem, the following was used:

```
* * * * * *
* * * * * *
* * * * * *   =36 desks
* * * * * *
* * * * * *
* * * * * *
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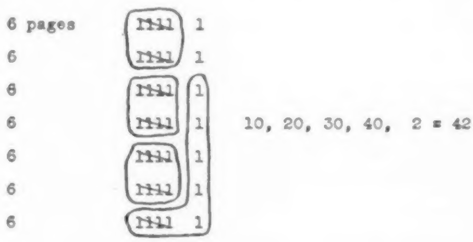
Others added a series of like numbers, as in the second problem, in this way:



These solutions were, of course, all recognized as correct. Attention was called to different solutions and to the interrelatedness of addition and counting as illustrated in the solutions. The fact that the third solution represented the true nature of the combination (making of 10's and 1's) of numbers was emphasized by the teacher.[3]

Some reasoned in this way (using the fourth problem as an example): "If four children brought six apples each, that would be 24. I already know that 6×4 is 24. Then 6×8 must be two times as much as that, or 48."

After the answers and various solutions had been discussed, the teacher asked for proof or methods of showing that certain facts, such as 6 9's are 54, are true. One boy offered this proof: To each of 6 children he gave 9 cards. Then he counted the cards. To his discomfort he didn't count to 54. "That's not what I got at my seat" was his remark. This demonstrated the superiority of the economical adult procedure over the longer, more easily understood counting procedure.

[1] Illustration of steps *a* and *b*.

[2] Illustration of step *c*.

[3] Illustration of step *d*.

The facts discovered or illustrated in the discussion were listed on the board. For example,

six 6's are 36 $6\times6=36$

six 7's are 42 $6\times7=42$

Later, of course, these facts were assembled in table form. This form shows relationships between facts better than any other arrangement.[4]

While the problems listed for this first day's work contained illustrations of multiplication only, division problems could have been used as well. In fact, on succeeding days the two types were used in the same lesson. In these later periods, longer, more cumbersome methods of accounting by ones and by groups (6's, 7's, 8's, or 9's), of adding, and of combinations of the above were accepted. Children were frequently asked to show how a certain combination of small groups of 1's could become a single group of 10's and 1's, or how a single group of 10's and 1's could be broken into certain specified smaller groups of 1's.

Division facts were placed on the board in this form:

$$\begin{array}{r} 6 \\ 7\overline{)42.} \\ \underline{42} \end{array}$$

The indicated operations in examples like $5/\overline{35}$ were brought out by the question, "How many 5's in 35?"

In addition to the multiplication and division tables placed on the board, each child made a table for himself. In connection with the multiplication table, the children were led to discover how closely counting by 6's, 7's, etc., was related to multiplying. This serial order of the multiplication facts was stressed frequently.

A combination of the two tables proved very popular with the children. This is shown as follows:

1	2	3	4	5	6	7	8	9
2	4	6	8	10	12	14	16	18
3	6	9	12	15	18	21	24	27
4	8	12	16	20	24	28	32	36
5	10	15	20	25	30	35	40	45
6	12	18	24	30	36	42	48	54
7	14	21	28	35	42	49	56	63
8	16	24	32	40	48	56	64	72
9	18	27	36	45	54	63	72	81

The children quickly saw the scheme used in building this square, namely, that the first row and first column represent numbers counted by 1's; that the second row and second column are counted by 2's, etc. As the children developed the square, they saw that a 24 was placed where they counted four 6's, and that this same 24 represents 6 counted by 4's. This reversal of the combinations gives the square an advantage over the usual tables of facts. The square was used later by individual children for purposes of study. For example, the first row and column were erased or covered, and then numbers on the chart selected at random. For each number selected, the child asked himself, "What two numbers multiplied together will give this number?" On other occasions, one row or column was left out and questions like "How many 6's in this number?" were asked. The square was also used for the solution of oral problems involving multiplication and division. Such oral activities made up a part of many of the arithmetic periods.[5]

However, it was recognized that eventually children need to know the facts automatically and that only part of them will learn these facts in a reasonable amount of time by use of the longer indirect methods alone. The procedures described above were used to provide for understanding, to give the children confidence, and to make possible the illustration of the need for

[4] Illustration of step *f*.

[5] Illustration of step *e*.

knowing the facts. After about three weeks in which the children were given opportunities to experiment in the discovery and proof of various facts, the following procedure was used.

At the beginning of one of the arithmetic periods the children were told to indicate the fact as soon as they had the answer to a problem and then to wait until others had it. "If one sleeps 9 hours a night for a week, how many hours of sleep has he had?" was the problem presented orally. The teacher noted the time required for children to solve the problem. Six children raised their hands immediately after the problem was stated. The other children required from 5 seconds to 4 minutes. This difference in working time was discussed. Then children who knew the answer instantaneously were asked how they attained it. Of course, the answer was that they just knew. Those who required more time told of using various indirect methods ranging from counting 9's to drawing marks and counting. In the discussion following this presentation of data on time and methods, the class quickly came to the conclusion that those who had learned the facts would be able to work faster, especially if a problem required several multiplication facts, or if there were several problems. It was also quite plain that it would be a nuisance always to have to "count up" or make a square to find the answer.[6]

The group was then asked if they could suggest anything which they might do to learn the facts for themselves. The following list was written on the board as the discussion progressed.

1. Practice counting by 2's, 3's, 4's, etc.
2. Write the tables in order from 2's through 9's.
3. Write the tables without answers in mixed order and try to get the correct solutions.
4. Use the multiplication square for checking answers to each of the above and for individual drill by telling how much 7 9's are, or 5 6's, or by telling what numbers can be multiplied to get 36 or 72, or by telling how many 8's are in 64, etc.
5. Practice by writing the answers to all the facts. (The teacher offered to put a list on the board without answers.)
6. Study the facts in card and table form by a method similar to that used in learning to spell a word.

The children were then told that from that time on it was their responsibility to learn the facts in the way or ways which they thought best. They were also told that it was a hard job which no one could do for them, but that the next few days would be set aside for direct study. When they were quite sure that they knew all the facts, they were allowed to come individually to the teacher for a quick flash card drill test. This individual test was given while other children were still working alone. After several days of study, a short test was given to all the group. Those who knew the facts were permitted to do advanced work. Those who did not had to spend more time in study, and a few were asked to go back to proving with objects each fact missed, or to finding out with objects those they did not know at all. Throughout the time of direct study, the teacher talked to the children individually to make sure that all had some plan of attack. A special effort was made to get each child to take advantage of relationships and not just repeat facts. No attempt was made, however, to dictate a method of study to the child. That the children had assumed the responsibility for learning the facts was stressed frequently.

Even after a week of direct study not all the children knew all the facts. However, every child knew enough to permit him to go on with more advanced work in multiplication and division. In this advanced work those who were not sure about facts were required to demonstrate those facts.[7] It was also suggested that they should again study these facts intensively.

[6] Illustration of step *g*.

[7] Illustration of step *d*.

◆ THE ART OF TEACHING ◆

Classroom Practice in the Teaching of Everyday Mathematics I

By Raymond J. Mejdak
Samuel E. Shull School, Perth Amboy, New Jersey

Probably the hardest thing for the teacher of Mathematics is to get the non-mathematical-minded pupils to think mathematically. The teacher of Algebra knows how difficult it is for pupils to transfer arithmetical thought into algebraic thought. Likewise the teacher of Geometry, in his attempt to develop a logical sequence of thought in the minds of pupils, encounters much difficulty. The teacher of Everyday Mathematics is not immune from similar trouble. The teachers of Mathematics are constantly endeavoring to instill into the minds of pupils the principles that make for the development of the powers of reasoning.

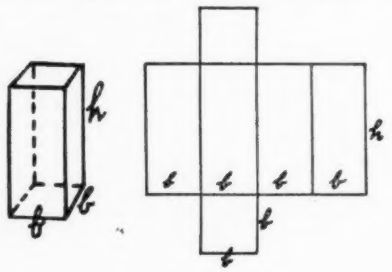


Fig. 1

Frequently the methods and devices for presentation or the tools and materials with which teachers must work are not altogether adequate. Because of this inadequacy, the presentation of some problems and situations for the pupils' understanding is often rendered abstract and vague.

The teacher who takes advantage of all situations by transforming the classroom into a mathematical laboratory with models, figures or apparatus, undoubtedly obtains better results from pupils than the teacher who resorts to an abstract exposition of the same problems.

In teaching the surface areas of solids, the teacher of Everyday Mathematics seizes the opportunity and uses actual models in the mathematical laboratory. To derive the formula of the entire surface of a square prism, $E.S. = 2b^2 + 4bh$, he uses a model and paper-form, cut to represent the entire surface.

By using a model he challenges the pupils to determine what constitutes the entire surface of a square prism. With the challenge the pupils see that the entire surface consists of two bases and a lateral or convex surface.

"What is the area of the square base? What is the area of the two bases? How is the area of the lateral surface found? How is the entire surface obtained?" These are questions that the pupils ask themselves to arrive at a solution.

Models are also used to derive the formulas for the entire surface of a rectangular prism, square and triangular pyramid, cube and cylinder.

The theory behind equations is taught by the use of a simple but real apparatus consisting of a ruler and a triangular solid or prism. The ruler is balanced at the 6-inch mark on the edge of the solid. The 6-inch mark corresponds to the equal sign in the equation. The ruler resembles the grocer's scale which is used in weighing.

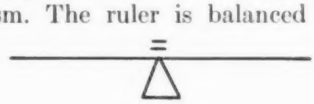


Fig. 2

By placing pennies on one side of the

ruler and equal weights on the other side to keep the balance, the pupils see how the equation works. Nothing is left to the imagination. After a few simple experiments the pupils are left with neither a vague nor an abstract idea of the relationship that exists between an equation and a grocer's scale.

In the study of addition and subtraction of signed numbers, a ruler divided into about forty divisions of positive and negative numbers and containing a sliding indicator or glass runner is used. By its use the theory of addition and subtraction of signed numbers is better understood. The pupils actually see what is done on the ruler when a signed number is added to or subtracted from another signed number.

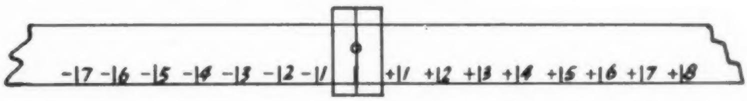


FIG. 3

If the teacher wishes to make the mathematics period interesting and worthwhile, he should transform it, whenever possible, into a sort of laboratory period where very useful experiments and problems may be worked with the use of a simple apparatus or object as a model.

EDITORIALS

The Unsolved Problem of Ninth-Year Mathematics
A Study and Appraisal of Present Curricular Trends

At the last annual meeting of the National Council of Teachers of Mathematics at Atlantic City on February 22, 1941, William Betz, Specialist in Mathematics for the Public Schools of Rochester, New York, spoke as follows:

Under the impact of conflicting curriculum theories and educational philosophies, the mathematical curriculum has become a scene of great confusion. At least twenty states have dropped all required work in secondary mathematics. In many schools the postponement movement has resulted in deferring algebra and geometry to the upper years of the high school. The courses are being made easier all along the line. Curriculum specialists are attempting to replace secondary mathematics by the arithmetic of common social backgrounds.

Reacting to the grave dangers of this development, important national organizations have recently issued sharp words of warning. Thus, at the Berkeley meetings of the Society for the Promotion of Engineering Education, in June, 1940, resolutions were passed which term it *suicidal*, at the present time, for our defense as a nation, *not to develop the most thorough kind of training for engineers*. It was declared to be "essential that a full four-year program of mathematics be available in the high schools for capable students, beginning with a year of college preparatory algebra in the *ninth grade*. This subject should *not* be postponed."[1] Again, the War Preparedness Committee of the American Mathematical Society and the Mathematical Association of America submitted to these organizations three very similar resolutions, which were adopted without delay. It was urged that these resolutions be given "*immediate publicity* in order that the changes recommended may be undertaken *at once*."

The three new national reports in the field of mathematics are likewise committed to a program of *systematic*, continuous training in mathematics.

Present indications are that there is developing a three-track system of general secondary mathematics, as follows:

1. A continuous course for competent students extending over *three or more years*, and offering substantial training in academic mathematics.
2. A course in general mathematics for non-academic students, largely for practical efficiency, extending over a period of *at least two years*.
3. A simplified, introductory course in general mathematics, for low-ability or retarded pupils.

Turning to a discussion of ninth-grade mathematics, Mr. Betz said that we now have at least eight types of curricular organization. Among these, the one-year terminal courses are of dubious value. A brief analysis of such courses was submitted. They differ widely in their offerings and in their aims. For slow pupils they usually attempt too much and for academic students they are really a waste of time. The speaker concluded that only a two-year or three-year sequence in general mathematics will produce the desired results for non-academic students.

It is high time that teachers of mathematics throughout the country give some attention to this matter. It is not possible in a course of one year to give an adequate foundation in mathematics to meet even the ordinary life needs of a well educated citizen, using the term "well educated" in the broad sense, to say nothing of preparing pupils for any possible future contacts with mathematics on the higher levels.

In this connection it is most undesirable that the idea gets around that general mathematics is only of significance for children of low ability in mathematics. While it may be true that such children find greater interest and subsequent happiness in such a course than in the older traditional approach it does not follow that the general mathematics progress *properly organized* is not equally desirable even if it has to be *more extensive* for the gifted pupils.

While mathematics is obviously of im-

[1] See also Bacon, H. M. "What is Happening to College Preparatory Mathematics," The Mathematics Teacher, XXXIV, 56–60.

portance in the defense program it is going to be of just as great importance in the days of peace which lie ahead. The mathematics that enters into the construction of military bombers is little if any different from that of the modern transport plane used in peace pursuits.

W. D. R.

Eighteenth Yearbook

DR. E. H. C. HILDEBRANDT of the State Teachers College of Upper Montclair, New Jersey, who is chairman of *The Multi-Sensory Aids Committee* of the National Council of Teachers of Mathematics which is preparing their final report as the Eighteenth Yearbook of the Council has sent us the following statement:

The Board of Directors of the National Council of Teachers of Mathematics voted at its annual convention at Atlantic City on February 22, 1941 that the Eighteenth Yearbook be devoted to a report on *The Use of Multi-Sensory Aids in the Teaching of Mathematics*. The Multi-Sensory Aids Committee was asked to begin plans on this report immediately. All material must be ready for final editing by May 1, 1942.

One of the functions of every yearbook is to summarize the work that has been done and is being carried on in the field of that particular study. Another purpose is to suggest future trends and problems for further experimentation and consideration.

Preliminary plans for the Eighteenth Yearbook call for:

1. Expository papers on various aids to mathematics teaching including such topics as history of mathematical models and aids, preparation and use of films, slides, mathematical instruments, construction and use of mathematical models in various units of study.
2. Short descriptions of models, devices, and other aids.
3. A complete bibliography of articles and sources.

Will every reader of THE MATHEMATICS TEACHER prepare *a list of the equipment and aids relating to mathematics and in use or available for use in his school?* The cooperation of all teachers is needed! Simply list the models used in your elementary, junior or senior high school or junior college. Do not hesitate to mention ordinary cardboard or wooden models. If a brief word of explanation or description, a diagram or a picture can be added, such will be welcomed. Pictures, films, film-slides, slides, charts and posters owned by the teacher or school should be added. Also commercial models such as geometric figures and solids, charts, model frames, slide rules, calculating instruments, graph boards, transits, sextants, etc. Do not assume that other teachers will send a similar list. The frequency with which certain models and aids are used is an important item in this study.

Send this list to E. H. C. Hildebrandt, Chairman of the Multi-Sensory Aids Committee of the National Council of Teachers of Mathematics, State Teachers College, Upper Montclair, New Jersey.

At the convention in Atlantic City, a preliminary bibliography on aids to mathematics teaching was distributed to all those in attendance. If you did not receive a copy, kindly indicate this when sending your list and a copy will be forwarded to you as long as the supply lasts.

It is hoped that teachers of mathematics throughout the country will respond to the above request. Any cooperative enterprise of this kind succeeds only to the extent to which teachers respond and respond promptly to such requests for their cooperation.

W. D. R.

A Merry Christmas and a Happy New Year!

IT MAY seem a little out of place to wish people a Merry Christmas and a Happy New Year at a time like this when it is certain that for most people such wishes cannot be fulfilled. However, THE MATHEMATICS TEACHER cannot let the opportunity go by without wishing all of its readers as merry a Christmas and as happy a New Year as possible under the circumstances.

W. D. R.

◆ IN OTHER PERIODICALS ◆

By Nathan Lazar
The Bronx High School of Science, New York City

The American Mathematical Monthly

June–July, 1941, Vol. 48, No. 6.

1. Moulton, E. J., "College Mathematics—a Statement for the Undergraduate," pp. 351–352.
2. Hart, William L., "On Education for Service," pp. 353–361.
3. Griffin, F. L., "War Mathematics" (A Syllabus of a Course), p. 362.
4. Hammer, P. C., "Plotting Curves in Polar Coördinates," p. 397.
5. Fry, Thornton C., "Industrial Mathematics," (Supplement to June–July, 1941 issue), pp. 1–38.

August–September, 1941, Vol. 48, No. 7.

1. Pólya, G., "Heuristic Reasoning and the Theory of Probability," pp. 450–465.
2. Kennedy, E. C., "A Note on Logarithms," pp. 465–467.
3. Stark, Marion E., "Constructions with Limited Means," pp. 475–479.

National Mathematics Magazine

October, 1941, Vol. 16, No. 1.

1. Sanders, S. T., "On Mathematical Certainty," pp. 2–6.
2. Yates, Robert C., "The Trisection Problem" (the fourth in a series of five chapters), pp. 20–28.
3. Williams, K. P., "The General Equation of the Second Degree," pp. 37–43.

School Science and Mathematics

October, 1941, Vol. 41, No. 7.

1. Neureiter, Paul R., "The Mathematics of War Bulletins," pp. 620–627.
2. Hoyt, John P., "Examples of Inexactness in an Exact Science," pp. 627–628.
3. Hewitt, Glenn F., "Mathematical Induction in High School Trigonometry," pp. 657–659

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Christmas meeting at
LEHIGH UNIVERSITY
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featuring:
An Exhibit
A Demonstration of Multi-sensory Aids
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 Senior High School Mathematics
 Junior High School Mathematics
 Teacher Training

Information—Inspiration—Association

For further particulars see the complete program on pages 289 and 290 of the November 1941 issue of The Mathematics Teacher.

BOOK REVIEWS

Elements of Aeronautics. By Francis Pope, Captain (First Pilot) with Transcontinental and Western Air, Inc., First Lieutenant United States Army Air Corps Reserve and Arthur S. Otis, Private Pilot, Fellow of the American Association for the Advancement of Science, Technical Member of the Institute of the Aeronautical Sciences, with an Introduction by Major Al Williams. World Book Company, Yonkers, New York. 1941. xii +653 pages. $3.40.

Both educational and aviation authorities have recognized the ever-increasing need for basic instruction for young people in this vitally important field. *Elements of Aeronautics* is an easy and practical approach to the science and art of aviation. It offers in one volume a complete introduction to aeronautics, organized for practical use by schools both large and small and written so that it is understood by students of high school age.

Elements of Aeronautics covers in an elementary way the art of flying, aerodynamics, air navigation, meteorology, and the rules and regulations all pilots must know.

In addition to supplying a much needed textbook for the general aeronautics course, it will interest teachers of mathematics and physics because of its wealth of applications which will make the principles more purposefully studied, more meaningful, and better remembered.

The elementary presentation of topics in this book should serve as a point of departure for a study of more technical books in any one of the various subdivisions of the subject. Here is thorough grounding in fundamental concepts—a sound basis for more specialized or technical reading. Because of the simplicity and thoroughness of its presentations and organization, it should make possible the immediate introduction of a course in aeronautics in any high school where such a course is desirable. W. D. R.

Basic Geometry. By George David Birkhoff and Ralph Beatley. Scott, Foresman and Company, 1941. 294 pp. Price, $1.32.

To say that his book is scholarly is akin to complimenting a prima donna on her singing. The names on the title page preclude the necessity of making such a statement. It is thrilling reading for one who is interested in the foundations of geometry and the role that postulates play in various systems.

It differs from other euclidean systems in that it assumes the number system and a one-to-one correspondence between the real numbers and the points on a line. (Euclid, of course, did not have the number system as a basis for his geometry.) By means of this attack, the authors are able to get to the heart of geometry much more quickly than previously. For example, the Pythagorean theorem is the seventh proved proposition.

The five assumptions on which the work is based are as follows:

Principle I. Line Measure. The points on a line can be numbered so that number differences measure distances.

Principle II. There is one and only one straight line through two given points.

Principle III. Angle Measure. All half-lines having the same end-point can be numbered so that number differences measure angles.

Principle IV. All straight angles have the same measure.

Principle V. Case I of Similarity. Two triangles are similar if an angle of one equals an angle of the other and the sides including the angle are proportional.

This last assumption is the one of course, that makes Basic Geometry a euclidean system. From this can be proved the usual postulate, Through a point outside a line only one line can be drawn parallel to a given line. This is proved as Proposition 13, the eighth proved proposition.

The concept of betweenness which Euclid tacitly assumed and which takes a good deal of abstract thinking along with several postulates and theorems in such a system as Veblen's geometry can be disposed of here very simply. For example, "Any point B lying on the line through the points A and C is said to be between A and C if the numbers corresponding to A, B, and C are in order."

The first proved proposition is the familiar one, coming usually in the second semester of study, "Two triangles are similar if two angles of one are equal to two angles of the other." Then almost immediately comes the theorem concerning the angle-sum of a traingle. The equality of triangles does not need to be taken as a separate topic as it can be considered as a special case of similarity. The word *congruence*, often implying superposition, is not used because "The logical foundation of our geometry is independent of any idea of motion." In constructions a marked ruler and a protractor are allowed as well as straightedge and compasses, but the limitations of the two usual tools are discussed.

To a teacher used to other textbooks in geometry, the sequence of propositions and the individual proofs may be disconcerting. It should be said, however, that pupils studying geometry for the first time will not have the same trouble that the teacher does.

The book could be improved as a high school text by more attention to those difficulties which it has been shown that pupils have with geometric figures and the meaning of proof. To be sure, the first chapter discusses the need of undefined and defined terms and assumptions and discusses the nature of both direct and indirect proof. Many apt non-mathematical illustrations are given in the hope of making these concepts understandable. But in the opinion of this reviewer, the psychology of the pupil is subordinated to the excellence of the geometric treatment. The one could be added without in any way impairing the other. However, "the proof of the pudding is in the eating" and no reviewer who has not used the book in the classroom has the right to criticize it from the teaching point of view. Before publication this text was used for seven years in the high school of Newton, Massachusetts, and for this reason its teachability is vouched for by the authors.

ROLLAND R. SMITH

Text and Tests in Elementary Algebra. By Smith, David Eugene, Reeve, William David and Morss, Edward L. Ginn and Company, Boston, 1941. 318 pages. Price $0.69, postage extra.

The first thing that an observer may note is the dignified title of the book, "Text and Tests in Elementary Algebra." The straight forward unaffectedness of this designation seems to be symbolic of the manner in which the authors have presented the subject.

One finds a good many evidences that the authors have taken into account the contributions which experience has made to the teaching of algebra. The order of topics is designed to give the student acquaintance with, and understanding of, algebraic symbolism well ahead of extended operations with algebraic numbers. In this particular especially the book seems to lessen the danger always present of promoting manipulation without comprehension.

There is a notable absence of material that is inconsistent with the best thought of the times, such as extended cases of factoring. On the other hand, the space thus saved has been used to early introduce scale drawings, statistics and indirect measurements. In the latter part of the text considerable attention is given to logarithms, the slide rule, and the organization of statistical data and central tendencies. It looks as though the provisions for saving time in the earlier part of the book might offer considerable promise of being able to reach some of these newer topics, without sacrificing an understanding of the general, and fundamental ideas.

There is a good index, and by following it through such topics as *numbers, graphs, formulas,* and the like, it is evident that the authors have formulated a very comprehensive outline of what algebra should comprise and that they have been especially successful in presenting relationship of ideas and application. This is apparent, for instance, in the way in which algebra, geometry, physics and economics are interwoven in the treatment of lever problems, roots, circular motion, and investments.

The exercises are judiciously chosen so as to provide the pupil with the advantages of drill while at the same time serving as illustrative material. The exercises seem to be arranged with reference to their importance, rather than from a pressing desire to fill out a page. This together with the number and grading of difficulty, which is apparent but not strained, should be of considerable assistance to the teacher presenting the subject so as to appeal to different levels of ability that generally exist in the same class.

The greatest advantages of general mathematics seem to be present in the manner in which topics are approached from the standpoint of making them understandable, rather than from a desire to illustrate or apply any particular phase of mathematics just for the sake of bringing it in.

The fact that the book contains the textbook subject matter of algebra should commend it to those who wish to be relieved of the necessity for purchasing both a text and a work-book. The exercises are generally presented with a view to giving the pupil an opportunity to realize that in working a problem he is dealing with the illustration of a principle—and that is another feature which should help to avoid meaningless manipulation.

The idea of dependence is adequately emphasized in a great number of situations, but one might wonder why the word function does not appear. Perhaps this is considered too hard a word. There is also the possibility that a question may be raised as to whether an attempt at simplification has not been carried too far in the statement that—"we multiply algebraic numbers just as we multiply numbers in arithmetic." Also fractions are presented without reference to their ratio meaning. However, these are minor considerations after all. The book reflects very positively the experience and scholarship of its authors.

JOHN P. EVERETT

Tools. A Mathematical Sketch and Model Book. By Robert C. Yates. Louisiana State University Press, Baton Rouge, 1941. 193 pages. $1.60.

This book is a distinct departure from the usual college geometry text. It is not a study of "modern geometry" nor a review of high school geometry; not a history of geometry nor a professional course for the training of teachers; yet all these features are embodied in it. The book has as its theme geometric tools. It begins with a presentation of the simple Euclidean tools, the unmarked straight edge and compasses, with discussions of their constructional possibilities. It progresses thence into a study of other modified and more or less complex geometrical instruments, including the parallel ruler, the angle ruler, quadratic and quartic tools, and a variety of plane linkages. Interspersed throughout are timely reviews of fundamental theorems, not only of Euclidean origin, but also from such sources as Menelaus, Ceva, Mascheroni, and Desargues. It is unmistakably the product of vast reading and research, coupled with careful organization and discrimination on the part of its author. It offers in one binder a variety of materials which previously one would have searched through many volumes and treatises to find. It offers in addition many new problems and new ideas of the author's own.

The book is in the form of a loose-leaf workbook and laboratory manual, making it convenient to transfer sheets to other standard loose-leaf binders, or to insert additional sheets. There are approximately 80 plates, containing about 500 drawings, each faced by explanatory text. Some of the drawings serve as problems, others as guides to the construction of cardboard models. Space is provided in the book itself for the student's work, to the end that the student will, in the words of the author, "develop the feeling of being co-author (of) a volume that may serve him later as a source of supplementary material in his career as a teacher." There are suggestions offered to the instructor as to the conduct of a course based upon these materials, such as a suitable division of time between classroom and laboratory.

No prerequisites beyond freshman college mathematics are listed, but it is obvious that the value of the book as a text "can be realized only by some thought and much labor" on the part of both student and instructor. This seems to imply the desirability of maturity and seriousness of purpose. Though the geometrical tool is the point of emphasis throughout, unquestionably the chief worth of the book lies in the providing of a clearer insight and deeper appreciation of geometrical thought and structure, which, in the mind of the writer, is the most desirable type of professional preparation.

TRYPHENA HOWARD

Mathematics, Its Magic and Mastery. By Aaron Bakst. D. Van Nostrand Company, Inc., 1941. 790 pp. $3.95.

This is a dangerous and seductive book, against which mathematics teachers should be warned, especially those weak souls, like myself, who cannot resist a numerical puzzle. It easily becomes a minor vice, seriously interfering with the performance of one's work. Take this review as an illustration. It was to have been ready for an earlier issue, but each time I put pen to paper to write it, I allowed myself one more glance into the book, happened upon a new puzzle I had not yet solved. At the end of an enchanted hour I discovered that no words had been written about Dr. Bakst's book, but I had solved a cryptogram, or found a new application for or illustration of a familiar mathematical principle, or had had some new ideas about the teaching of mathematics. Now the review must be written and I have decided to remove temptation by locking the subject of the review firmly in the lower drawer of my desk, putting the key in a not-too-convenient spot, and discussing it from memory.

The casual reader looking over the table of contents with its gaily worded subheadings (How to multiply, and like it; The other side of zero; It's hotter than you think; Debt on the installment plan; The rich get richer and the poor try to borrow; Every number has its fingerprint; How to make money breed; Hollywood goes mathematical; Meet the circle's fat friends) may fail to get any clear idea as to the amount or difficulty of the mathematical topics treated in the book. The sixty-six pages of straight mathematical text in the appendix deals with such algebraic topics as signed numbers, multiplication of polynomials, treatment of algebraic fractions, powers and roots, square root of numbers, arithmetic and geometric progressions, logarithms, equations of first and second degree, and deals with elementary geometric figures including three dimensional figures. No proofs are given, but many simple geometric and trigonometric relations are stated for reference. Tables of mantissas, squares and square roots, sine, cosine and tangent are provided. There is a very useful page of fourteen approximate formulas for simplified computation. The fact that the list of answers to problems in the text covers eleven pages gives a fairly good idea of the amount of problem material covered.

The New York Times Magazine Section for November 9, 1941, contains an article on "Codes

Used by Spies," which says, "The code message thrives because so few persons think in terms of cryptograms." This book devotes over twenty pages to code-writing. Among other topics not found in the ordinary text are rapid calculation, number systems with basis other than 10, e (which he calls "the banker's number"), finger reckoning, formula for continuous growth, degrees of freedom, flatland, n-dimensional figures. There are some excellent verbal problems for the teacher of elementary mathematics who is weary of the standard problems.

The strength of the book lies more in the variety and richness of the spicy illustrations and applications which Dr. Bakst has so tirelessly amassed than in completeness or clarity of exposition. Almost every page contains some element of the unexpected, the curious. Yet almost all of these unorthodox problems are based on mathematics which is not intrinsically too difficult for a bright twelfth grade pupil and many could be used effectively with a ninth grade class. The applications are drawn from the circus, astronomy, physics, surveying, ballistics, finançe, engineering, aeronautics, games of chances, carpentry, installment buying, insurance, highway accidents, and many other sources, as well as from sheer, playful nonsense.

HELEN M. WALKER

Problem

Simplify:

$$\frac{e^{2/3}r^2y}{M^{-1}e^{-1/3}} \cdot \frac{\sqrt{x}M^2a}{x^{-1/2}MS^{-1}}$$

Do you get it?


CAN I
HIM FOR
TMAS BESIDES
S, TIES,
?
WHAT CAN I
GET HER FOR
CHRISTMAS
BESIDES A
COMPACT,
PERFUM
MAY I
SUGGEST
A
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THE MATHEMATICS TEACHER

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